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Inverse scattering transform and multiple high-order pole solutions for the Gerdjikov–Ivanov equation under the zero/nonzero background

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Abstract. In this article, the inverse scattering transform is considered for the Gerdjikov–Ivanov equation with zero and nonzero boundary conditions by a matrix Riemann–Hilbert (RH) method. The formula of the soliton solutions is established by Laurent expansion to the RH problem. The method we used is different from computing solution with simple poles since the residue conditions here are hard to be obtained. The formula of multiple soliton solutions with one high-order pole and N multiple high-order poles are obtained, respectively. The dynamical properties and characteristic for the high-order pole solutions are further analyzed.

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Keywords. Gerdjikov–Ivanov equation, Riemann–Hilbert problem, Multiple high-order poles, Soliton solution.

Contents

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1. Introduction

The inverse scattering transform method plays a significant role during the discovery process of the exact solutions of completely integrable systems [1,2]. As a new version of inverse scattering transform method, the Riemann–Hilbert (RH) approach has become the preferred research technique to the researchers in investigating the soliton solutions and the long-time asymptotics of integrable systems in recent years [3,4]. More recently, the RH approach has been widely used to investigate the integrable systems with nonzero boundary [5–10]. The high-order soliton solutions which have the same velocity and locate at the same position also have been studied [11–13]. The general method to obtain the high-order solitons with the classical inverse scattering transform (IST) method involves some complicated calculation, especially for the case of multiple high-order poles [14–16]. In this case, it is effective to construct high-order pole soliton solutions of integrable systems by Laurent expansion to the RH problem [16–18].

It is well-known that the nonlinear Schrödinger (NLS) equation [19,20]

$$iq_t + q_{xx} + 2|q|^2 q = 0 \tag{1.1}$$

is one of the most important integrable systems, which plays an important role and has applications in a wide variety of fields. Besides the NLS equation (1.1), derivative NLS (DNLS) equations were also introduced to investigate the effects of high-order perturbations [21–23]. Among them, there are three derivative NLS equations [23], the first one is Kaup–Newell equation [24]

$$iq_t + q_{xx} + i(|q|^2)_x = 0. \tag{1.2}$$

The second type is the Chen–Lee–Liu equation [25]

$$iq_t + q_{xx} + i|q|^2 q_x = 0. \tag{1.3}$$

The third type is the Gerdjikov–Ivanov (GI) equation which takes the form [26]

$$iq_t + q_{xx} - iq^2 q_x^* + \frac{1}{2} q^3 q^{*2} = 0, \tag{1.4}$$

where the asterisk $*$ means the complex conjugation. The DNLS equations are regarded as models in a wide variety of fields such as weakly nonlinear dispersive water waves, nonlinear optical fibers, quantum field theory and plasmas [27–30]. In plasma physics, the GI equation (1.4) is a model for Alfv*é*n waves propagating parallel to the ambient magnetic field, where q being the transverse magnetic field perturbation and x and t being space and time coordinates, respectively [31,32]. The GI equation has been studied through many methods. For instance, the Darboux transformation [33], the nonlinearization [34,35], the similarity reduction, the bifurcation theory and others [36,37]. Especially, RH method is used to construct N-soliton of the GI equation with zero boundary [38]. Recently, we used the RH method to construct simple pole solutions of the GI equation with nonzero boundary conditions [39].

In this article, we further investigate the inverse scattering transform and high-order solutions of GI equation (1.4) with zero boundary condition

$$q(x,t) \to 0, \qquad x \to \pm\infty, \tag{1.5}$$

and the following nonzero boundary conditions

$$q(x,t) \sim q_{\pm} e^{-\frac{3}{2} i q_0^4 t + i q_0^2 x}, \qquad x \to \pm\infty, \tag{1.6}$$

where $|q_{\pm}| = q_0 > 0$, and $q_{\pm}$ are independent of x, t. The formula of multiple soliton solutions of the GI equation is obtained, which correspond to multiple high-order poles of the RH problem.

This paper is organized as follows. In Sect. 2, we construct the RH problem of GI equation (1.4) with zero boundary condition and display the relationship between the solutions of the RH problem and GI equation, then we derive the formula of single high-order solutions and multiple high-order solutions of GI equation. In Sect. 3, by the same method, we give the one single high-order soliton solution and multiple high-order soliton solutions of GI equation (1.4) with nonzero boundary condition. We give the patterns for both zero and nonzero boundary conditions.

2. IST with zero boundary and high-order pole

2.1. Spectral analysis

2.1.1. Eigenfunctions and scattering matrix. It is well-known that the GI equation (1.4) admits the Lax pair [33]

$$\psi_x = X\psi, \qquad \psi_t = T\psi, \tag{2.1}$$

where

$$X = -ik^2\sigma_3 + kQ - \frac{i}{2}Q^2\sigma_3, \tag{2.2}$$

$$T = -2ik^4\sigma_3 + 2k^3Q - ik^2Q^2\sigma_3 - ikQ_x\sigma_3 + \frac{1}{2}(Q_xQ - QQ_x) + \frac{i}{4}Q^4\sigma_3, \tag{2.3}$$

and

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}. \tag{2.4}$$

With zero boundary (1.5), asymptotic spectra problem of the Lax pair (2.1) becomes

$$\psi_x = X_\pm\psi, \qquad \psi_t = T_\pm\psi, \tag{2.5}$$

where

$$X_\pm = -ik^2\sigma_3, \qquad T_\pm = -2ik^3\sigma_3. \tag{2.6}$$

We define the Jost eigenfunctions $\phi_\pm(x,t,k)$ as the simultaneous solutions of both parts of the Lax pair, so that

$$\phi(x,t,k) = \psi(x,t,k)e^{i\theta(k)\sigma_3}, \tag{2.7}$$

where $\theta(k) = k^2(x + 2k^2t)$, then

$$\phi_\pm(x,t,k) \to I, \qquad x \to \pm\infty. \tag{2.8}$$

Meanwhile, $\phi_\pm$ acquire the equivalent Lax pair

$$\phi_x(x,t,k) + ik^2[\sigma_3, \phi(x,t,k)] = \Delta X_\pm\phi(x,t,k); \tag{2.9a}$$

$$\phi_t(x,t,k) + 2ik^4[\sigma_3, \phi(x,t,k)] = \Delta T_\pm\phi(x,t,k), \tag{2.9b}$$

where $\Delta X_\pm = X - X_\pm$ and $\Delta T_\pm = T - T_\pm$.

Since $\psi_\pm(x,t,k)$ are two fundamental matrix solutions, there exists a constant matrix $S(k)$ such that

$$\psi_+(x,t,k) = \psi_-(x,t,k)S(k), \tag{2.10}$$

where $S(k) = (s_{ij}(k))_{2\times 2}$ is referred to the scattering matrix and its entries as the scattering coefficients. It follows from (2.10) that s_{ij} has the Wronskian representation:

$$s_{11}(k) = \mathrm{Wr}(\psi_{+,1}, \psi_{-,2}), \qquad s_{12}(k) = \mathrm{Wr}(\psi_{+,2}, \psi_{-,2}), \tag{2.11a}$$

$$s_{21}(k) = \mathrm{Wr}(\psi_{-,1}, \psi_{+,1}), \qquad s_{22}(k) = \mathrm{Wr}(\psi_{-,1}, \psi_{+,2}). \tag{2.11b}$$

2.1.2. Analyticity. As a result, the Volterra integral equations are

$$\phi_{\pm}(x,t,k) = I + \int_{\pm\infty}^{x} e^{-ik^2(x-y)\sigma_3} \Delta X_{\pm}\phi_{\pm}(y,t,k) e^{ik^2(x-y)\sigma_3} \mathrm{d}y. \tag{2.12}$$

We define D^+, D^- and Σ as

$$D^+ := \{k \in \mathbb{C} : \mathrm{Re}k\mathrm{Im}k > 0\}, \quad D^- := \{k \in \mathbb{C} : \mathrm{Re}k\mathrm{Im}k < 0\}, \quad \Sigma := \mathbb{R} \cup i\mathbb{R}. \tag{2.13}$$

Proposition 1. *Suppose that $q(x,t) \in L^1(\mathbb{R})$ and $\phi_{\pm,j}(x,t,k)$ denotes the jth column of $\phi_{\pm}(x,t,k)$, then $\phi_{\pm}(x,t,k)$ have the following properties:*

- *$\phi_{-,1}$,$\phi_{+,2}$ and s_{22} are analytic in D^+ and continuous in $D^+ \cup \Sigma$.*
- *$\phi_{+,1}$, $\phi_{-,2}$ and s_{11} are analytic in D^- and continuous in $D^- \cup \Sigma$.*
- *s_{12} and s_{21} are continuous on Σ.*

As usual, the reflection coefficients $r(k)$ are defined as

$$r(k) = \frac{s_{12}(k)}{s_{22}(k)}, \qquad \tilde{r}(k) = \frac{s_{21}(k)}{s_{11}(k)}, \qquad k \in \Sigma. \tag{2.14}$$

2.1.3. Symmetries.

Proposition 2. *The Jost solution, scattering matrix and reflection coefficients satisfy the following reduction conditions*

- *The first symmetry reduction*

$$\phi_{\pm}(x,t,k) = \sigma_2\phi_{\pm}^*(x,t,k^*)\sigma_2, \qquad S(k) = \sigma_2 S(k^*)^*\sigma_2, \qquad r(k) = -\tilde{r}(k^*)^*, \tag{2.15}$$

- *The second symmetry reduction*

$$\phi_{\pm}(x,t,k) = \sigma_1\phi_{\pm}^*(x,t,-k^*)\sigma_1, \qquad S(k) = \sigma_1 S(-k^*)^*\sigma_1, \qquad r(k) = \tilde{r}(-k^*)^*, \tag{2.16}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{2.17}$$

2.1.4. Asymptotic behaviors. To solve the RH problem in the next section, it is necessary to discuss the asymptotic behaviors of the modified Jost solutions and scattering matrix as $k \to \infty$ by the standard Wentzel–Kramers–Brillouin (WKB) expansions.

Proposition 3. *The asymptotic behaviors for the modified Jost solutions and scattering matrix are given as*

$$\phi_{\pm}(x,t,k) = I - \frac{i}{2k}\sigma_3 Q + o(k^{-1}), \qquad S(k) = I + o(k^{-1}), \qquad k \to \infty \tag{2.18}$$

Furthermore, solutions of the GI equation will be constructed by

$$q(x,t) = \lim_{k\to\infty} 2i(k\phi_{\pm})_{12}. \tag{2.19}$$

2.2. Riemann–Hilbert problem

As we all know, the equation (2.10) is the beginning of the formulation of the inverse problem. We always regard it as a relation between eigenfunctions analytic in D^+ and those analytic in D^-. Thus, it is necessary for us to introduce the following RH problem.

Proposition 4. *Define the sectionally meromorphic matrix*

$$M(x,t,k) = \begin{cases} M^- = \begin{pmatrix} \dfrac{\phi_{+,1}}{s_{11}} & \phi_{-,2} \end{pmatrix}, \text{ as } k \in D^-, \\ M^+ = \begin{pmatrix} \phi_{-,1} & \dfrac{\phi_{+,2}}{s_{22}} \end{pmatrix}, \text{ as } k \in D^+. \end{cases} \tag{2.20}$$

Then a multiplicative matrix RH problem is proposed:

- *Analyticity: $M(x,t,k)$ is analytic in $\mathbb{C} \setminus \Sigma$.*
- *Jump condition*

$$M^-(x,t,k) = M^+(x,t,k)(I - G(x,t,k)), \qquad k \in \Sigma, \tag{2.21}$$

where

$$G(x,t,k) = \begin{pmatrix} r(k)\tilde{r}(k) & e^{2i\theta}\tilde{r}(k) \\ -e^{-2i\theta}r(k) & 0 \end{pmatrix}. \tag{2.22}$$

- *Asymptotic behaviors*

$$M(x,t,k) \sim I + O(k^{-1}), \qquad k \to \infty, \tag{2.23}$$

Moreover, new solutions of the GI equation can be reconstructed by $M(x,t,k)$ as

$$q(x,t) = \lim_{k\to\infty} 2i(kM)_{12}. \tag{2.24}$$

2.3. Single high-order pole solutions

We assume $s_{22}(k)$ have high-order poles $\{\pm k_j : \operatorname{Re} k_j > 0\}_{j=1}^N$, from the symmetries (2.15) and (2.16), we know that $\{\pm k_j^* : \operatorname{Im} k_j^* < 0\}_{j=1}^N$ are high-order poles of $s_{11}(k)$. So $s_{22}(k)$ can be expanded as:

$$s_{22}(k) = (k^2 - k_1^2)^{n_1}(k^2 - k_2^2)^{n_2} \cdots (k^2 - k_N)^{n_N} s_0(k), \tag{2.25}$$

where $s_0(k) \neq 0$ for all $k \in D^+$. When $s_{22}(k)$ only has N simple zeros, the RHP can be solved straightforward by the residue conditions, and the formula of Nth-order soliton solutions of GI equation are obtained through (2.24). However, as $s_{22}(k)$ has multiple high-order zero points, the residue conditions are not enough, and the coefficients related to much higher negative power of $k \pm k_j$ and $k \pm k_j^*$ should be considered. For convenience, we will consider the simplest case at first where $s_{22}(k)$ has only one higher-order zero point.

Let $k_0 \in D^+$ be the Nth-order pole, from the symmetries (2.15) and (2.16) it is obvious that $-k_0 \in D^+$ also is the Nth-order pole of $s_{22}(k)$. Then $\pm k_0^*$ are the Nth-order poles of $s_{11}(k)$. The discrete spectrum is the set

$$\{\pm k_0, \pm k_0^*\}, \tag{2.26}$$

which can be seen in Fig. 1.

Let

$$s_{22}(k) = (k^2 - k_0^2)s_0(k), \tag{2.27}$$

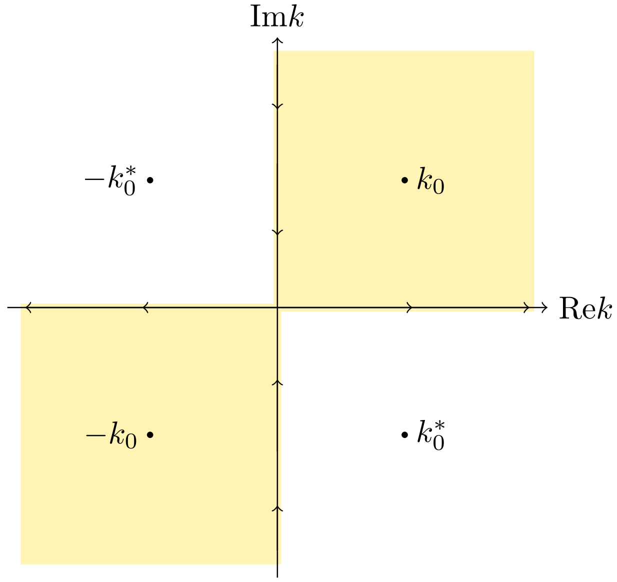

FIG. 1. Distribution of the the discrete spectrum and the contours for the RH problem on complex k-plane

in which $s_0(k) \neq 0$ in D^+. According to the Laurent series expansion in poles, $r(k)$ and $r^*(k^*)$ can be, respectively, expanded as

$$r(k) = r_0(k) + \sum_{m=1}^{N} \frac{r_m}{(k-k_0)^m}, \text{ in } k_0; \qquad r(k) = \tilde{r}_0(k) + \sum_{m=1}^{N} \frac{(-1)^{m+1} r_m}{(k+k_0)^m} \text{ in } -k_0; \tag{2.28a}$$

$$r^*(k^*) = r_0^*(k^*) + \sum_{m=1}^{N} \frac{r_m^*}{(k-k_0^*)^m} \text{ in } k_0^*; \qquad r^*(k^*) = \tilde{r}_0^*(k^*) + \sum_{m=1}^{N} \frac{(-1)^{m+1} r_m^*}{(k+k_0^*)} \text{ in } -k_0^*; \tag{2.28b}$$

where r_m are defined by

$$r_m = \lim_{k \to k_0} \frac{1}{(N-m)!} \frac{\partial^{N-m}}{\partial k^{N-m}} [(k-k_0)^N r(k)], \qquad m = 1, 2, \ldots, N, \tag{2.29}$$

and $r_0(k)$ and $\tilde{r}_0(k)$ are analytic for all $k \in D^+$. The definition of $M(x,t,k)$ yields that $k = \pm k_0$ are Nth-order poles of M_{12}, while $k = \pm k_0^*$ are Nth-order poles of M_{11}. According to the normalization condition sated in proposition 5, one can set

$$M_{11}(x,t,k) = 1 + \sum_{s=1}^{N} \Big(\frac{F_s(x,t)}{(k-k_0^*)^s} + \frac{H_s(x,t)}{(k+k_0^*)^s} \Big), \tag{2.30a}$$

$$M_{12}(x,t,k) = \sum_{s=1}^{N} \Big(\frac{G_s(x,t)}{(k-k_0)^s} + \frac{L_s(x,t)}{(k+k_0)^s} \Big), \tag{2.30b}$$

where $F_s(x,t)$, $H_s(x,t)$, $G_s(x,t)$, $L_s(x,t)(s = 1, 2, \ldots, N)$ are unknown functions which need to be determined. Once these functions are solved, the solution $M(x,t,k)$ of RHP will be obtained and the solutions $q(x,t)$ of the GI equation will be obtained from (2.24).

Now we are in position to solve $F_s(x,t)$, $H_s(x,t)$, $G_s(x,t)$ and $L_s(x,t)(s=1,2,\ldots,N)$. According to Taylor series expansion, one has

$$e^{-2i\theta(k)}=\sum_{l=0}^{+\infty}f_l(x,t)(k-k_0)^l,\qquad e^{-2i\theta(k)}=\sum_{l=0}^{+\infty}(-1)^l f_l(x,t)(k+k_0)^l, \tag{2.31a}$$

$$e^{2i\theta(k)}=\sum_{l=0}^{+\infty}f_l^*(x,t)(k-k_0^*)^l,\qquad e^{2i\theta(k)}=\sum_{l=0}^{+\infty}(-1)^l f_l^*(x,t)(k+k_0^*)^l, \tag{2.31b}$$

$$M_{11}(x,t,k)=\sum_{l=0}^{+\infty}\mu_l(x,t)(k-k_0)^l,\qquad M_{11}(x,t,k)=\sum_{l=0}^{+\infty}(-1)^l\mu_l(x,t)(k+k_0)^l, \tag{2.31c}$$

$$M_{12}(x,t,k)=\sum_{l=0}^{+\infty}\zeta_l(x,t)(k-k_0^*)^l,\qquad M_{12}(x,t,k)=\sum_{l=0}^{+\infty}(-1)^{l+1}\zeta_l(x,t)(k+k_0^*)^l, \tag{2.31d}$$

where

$$f_l(x,t)=\lim_{k\to k_0}\frac{1}{l!}\frac{\partial^l}{\partial k^l}e^{-2ik^2(x+2k^2t)}; \tag{2.32a}$$

$$\mu_l(x,t)=\lim_{k\to k_0}\frac{1}{l!}\frac{\partial^l}{\partial k^l}M_{11}(x,t,k),\qquad \zeta_l(x,t)=\lim_{k\to k_0^*}\frac{1}{l!}\frac{\partial^l}{\partial k^l}M_{12}(x,t,k). \tag{2.32b}$$

When $k\in D^+$, we have the expansions in $k=k_0$

$$M_{11}(k)=\phi_{-,11}=\sum_{l=0}^{+\infty}\mu_l(x,t)(k-k_0)^l,\qquad M_{12}(k)=\frac{\phi_{+,22}}{s_{22}}=e^{-2i\theta}r(k)\phi_{-,11}+\phi_{-,12}. \tag{2.33}$$

comparing the coefficients of $(k-k_0)^{-s}$ with (2.30b), we can get

$$G_s(x,t)=\sum_{j=s}^{N}\sum_{l=0}^{j-s}r_j f_{j-s-l}(x,t)\mu_l(x,t). \tag{2.34}$$

Similarly, from the expansions in $k=-k_0$, we can get that

$$L_s(x,t)=\sum_{j=s}^{N}\sum_{l=0}^{j-s}(-1)^{s+1}r_j f_{j-s-l}(x,t)\mu_l(x,t). \tag{2.35}$$

With the same method, when $k\in D^-$, we can obtain that

$$F_s(x,t)=-\sum_{j=s}^{N}\sum_{l=0}^{j-s}r_j^* f_{j-s-l}^*(x,t)\zeta_l(x,t),\qquad H_s(x,t)=\sum_{j=s}^{N}\sum_{l=0}^{j-s}(-1)^{s+1}r_j^* f_{j-s-l}^*(x,t)\zeta_l(x,t). \tag{2.36}$$

Actually, $\mu_l(x,t)$ and $\zeta_l(x,t)$ can also be expressed by $F_s(x,t)$, $H_s(x,t)$, $G_s(x,t)$ and $L_s(x,t)$. Recalling the definitions of $\zeta_l(x,t)$ and $\mu_l(x,t)$ given by (2.32b) and substituting (2.30) into them, we can obtain

$$\zeta_l(x,t)=\sum_{s=1}^{N}\binom{s+l-1}{l}\left\{\frac{(-1)^l G_s(x,t)}{(k_0^*-k_0)^{l+s}}+\frac{(-1)^l L_s(x,t)}{(k_0^*+k_0)^{l+s}}\right\},\qquad l=0,1,2,\ldots, \tag{2.37}$$

$$\mu_l(x,t)=\begin{cases}1+\sum\limits_{s=1}^{N}\left\{\frac{F_s(x,t)}{(k_0-k_0^*)^s}+\frac{H_s(x,t)}{(k_0+k_0^*)^s}\right\}, & l=0;\\ \sum\limits_{s=1}^{N}\binom{s+l-1}{l}\left\{\frac{(-1)^l F_s(x,t)}{(k_0-k_0^*)^{s+l}}+\frac{(-1)^l H_s(x,t)}{(k_0+k_0^*)^{s+l}}\right\}, & l=1,2,3,\ldots\end{cases} \tag{2.38}$$

Using (2.36) and (2.3), we obtain the system

$$F_s(x,t) = -\sum_{j=s}^{N}\sum_{l=0}^{j-s}\sum_{p=1}^{N}\binom{p+l-1}{l} r_j^* f_{j-s-l}^* \left\{ \frac{(-1)^l G_p(x,t)}{(k_0^*-k_0)^{l+p}} + \frac{(-1)^l L_p(x,t)}{(k_0^*+k_0)^{l+p}} \right\}, \tag{2.39a}$$

$$H_s(x,t) = \sum_{j=s}^{N}\sum_{l=0}^{j-s}\sum_{p=1}^{N}(-1)^{s+1}\binom{p+l-1}{l} r_j^* f_{j-s-l}^* \left\{ \frac{(-1)^l G_p(x,t)}{(k_0^*-k_0)^{l+p}} + \frac{(-1)^l L_p(x,t)}{(k_0^*+k_0)^{l+p}} \right\}, \tag{2.39b}$$

$$G_s(x,t) = \sum_{j=s}^{N} r_j f_{j-s} + \sum_{j=s}^{N}\sum_{l=0}^{j-s}\sum_{p=1}^{N}\binom{p+l-1}{l} r_j f_{j-s-l} \left\{ \frac{(-1)^l F_p(x,t)}{(k_0-k_0^*)^{l+p}} + \frac{(-1)^l H_p(x,t)}{(k_0+k_0^*)^{l+p}} \right\}, \tag{2.39c}$$

$$\begin{aligned} L_s(x,t) &= \sum_{j=s}^{N}(-1)^{s+1} r_j f_{j-s} \\ &\quad + \sum_{j=s}^{N}\sum_{l=0}^{j-s}\sum_{p=1}^{N}(-1)^{s+1}\binom{p+l-1}{l} r_j f_{j-s-l} \left\{ \frac{(-1)^l F_p(x,t)}{(k_0-k_0^*)^{l+p}} + \frac{(-1)^l H_p(x,t)}{(k_0+k_0^*)^{l+p}} \right\}, \end{aligned} \tag{2.39d}$$

Let us introduce

$$|\eta\rangle = (\eta_1,\ldots,\eta_N)^T, \quad \eta_s = \sum_{j=s}^{N} r_j f_{j-s}(x,t), \tag{2.40}$$

$$|\tilde{\eta}\rangle = (\tilde{\eta}_1,\ldots,\tilde{\eta}_N)^T, \quad \tilde{\eta}_s = \sum_{j=s}^{N}(-1)^{s+1} r_j f_{j-s}(x,t), \tag{2.41}$$

$$|F\rangle = (F_1,F_2,\ldots,F_N)^T, \qquad |H\rangle = (H_1,H_2,\ldots,H_N)^T, \tag{2.42}$$

$$|G\rangle = (G_1,G_2,\ldots,G_N)^T, \qquad |L\rangle = (L_1,L_2,\ldots,L_N)^T, \tag{2.43}$$

$$\Omega_1 = [\Omega_{1,sp}]_{N\times N} = \Big[-\sum_{j=s}^{N}\sum_{l=0}^{j-s}\binom{p+l-1}{l}\frac{(-1)^l r_j^* f_{j-s-l}^*(x,t)}{(k_0^*-k_0)^{l+p}} \Big]_{N\times N}, \tag{2.44}$$

$$\Omega_2 = [\Omega_{2,sp}]_{N\times N} = \Big[-\sum_{j=s}^{N}\sum_{l=0}^{j-s}\binom{p+l-1}{l}\frac{(-1)^l r_j^* f_{j-s-l}^*(x,t)}{(k_0^*+k_0)^{l+p}} \Big]_{N\times N}, \tag{2.45}$$

$$\Omega_3 = [\Omega_{3,sp}]_{N\times N} = \Big[\sum_{j=s}^{N}\sum_{l=0}^{j-s}\binom{p+l-1}{l}\frac{(-1)^{s+l+1} r_j^* f_{j-s-l}^*(x,t)}{(k_0^*-k_0)^{l+p}} \Big]_{N\times N}, \tag{2.46}$$

$$\Omega_4 = [\Omega_{4,sp}]_{N\times N} = \Big[\sum_{j=s}^{N}\sum_{l=0}^{j-s}\binom{p+l-1}{l}\frac{(-1)^{s+l+1} r_j^* f_{j-s-l}^*(x,t)}{(k_0^*+k_0)^{l+p}} \Big]_{N\times N}, \tag{2.47}$$

where the superscript ^{T} denotes the transposed matrix. Thus, the linear system (2.39) can be rewritten as

$$\begin{cases} I|F\rangle + \mathbf{0}|H\rangle - \Omega_1|G\rangle - \Omega_2|L\rangle = \mathbf{0} \\ \mathbf{0}|F\rangle + I|H\rangle - \Omega_3|G\rangle - \Omega_4|L\rangle = \mathbf{0} \\ \Omega_1^*|F\rangle + \Omega_2^*|H\rangle + I|G\rangle + \mathbf{0}|L\rangle = |\eta\rangle \\ \Omega_3^*|F\rangle + \Omega_4^*|H\rangle - \mathbf{0}|G\rangle - I|L\rangle = -|\tilde{\eta}\rangle \end{cases} \tag{2.48}$$

Through direct calculations, $(|F\rangle,|H\rangle)^T$ and $(|G\rangle,|L\rangle)^T$ are explicitly solved as

$$(|F\rangle,|H\rangle)^T = \Omega(I_\sigma+\Omega^*\Omega)^{-1})(|\eta\rangle,-|\tilde{\eta}\rangle)^T, \tag{2.49}$$

$$(|G\rangle,|L\rangle)^T = (I_\sigma+\Omega^*\Omega)^{-1})(|\eta\rangle,-|\tilde{\eta}\rangle)^T, \tag{2.50}$$

where

$$\Omega=\begin{pmatrix}\Omega_1 & \Omega_2\\ \Omega_3 & \Omega_4\end{pmatrix},\qquad I_\sigma=\begin{pmatrix}I_{N\times N} & \\ & -I_{N\times N}\end{pmatrix}. \tag{2.51}$$

Substituting $(|F\rangle,|H\rangle)^T$ and $(|G\rangle,|L\rangle)^T$ into the expansions of $M_{11}(x,t,k)$ and $M_{12}(x,t,k)$ given by (2.30), since it is well known that for matrix $A_{m\times n}$, $B_{n\times m}\in\mathbb{K}$ ($\mathbb{K}$ is a field of numbers) $\det(I_m+AB)=\det(I_n+BA)$ (I_m and I_n are m and n dimension identity matrix, respectively), we get that

$$\begin{aligned}M_{11}(x,t,k)&=1+\big(\langle Y|,\langle\tilde{Y}|)(|F\rangle,|H\rangle\big)^T\\&=\det\big(1+(\langle Y|,\langle\tilde{Y}|)(|F\rangle,|H\rangle)^T\big)\\&=\det\big(1+(\langle Y|,\langle\tilde{Y}|)\Omega(I_\sigma+\Omega^*\Omega)^{-1}(|\eta\rangle,-|\tilde{\eta}\rangle)^T\big)\\&=\det\big(I+(|\eta\rangle,-|\tilde{\eta}\rangle)^T(\langle Y|,\langle\tilde{Y}|)\Omega(I_\sigma+\Omega^*\Omega)^{-1}\big)\\&=\frac{\det\big(I_\sigma+\Omega^*\Omega+(|\eta\rangle,-|\tilde{\eta}\rangle)^T(\langle Y|,\langle\tilde{Y}|)\Omega\big)}{\det\big(I_\sigma+\Omega^*\Omega\big)},\end{aligned} \tag{2.52}$$

where

$$\langle Y(k)|=\Big(\frac{1}{k-k_0^*},\frac{1}{(k-k_0^*)^2},\ldots,\frac{1}{(k-k_0^*)^N}\Big),\quad \langle\tilde{Y}(k)|=\Big(\frac{1}{k+k_0^*},\frac{1}{(k+k_0^*)^2},\ldots,\frac{1}{(k+k_0^*)^N}\Big). \tag{2.53}$$

In the same way, we get that

$$M_{12}=\frac{\det\Big(I_\sigma+\Omega^*\Omega+(|\eta\rangle,-|\tilde{\eta}\rangle)^T(\langle Y^*(k^*)|,\langle\tilde{Y}^*(k^*)|)\Big)}{\det\big(I_\sigma+\Omega^*\Omega\big)}-1. \tag{2.54}$$

Theorem 1. *With the rapidly decaying initial condition (1.5), the Nth order soliton of GI equation is*

$$q(x,t)=2i\left[\frac{\det\Big(I_\sigma+\Omega^*\Omega+(|\eta\rangle,-|\tilde{\eta}\rangle)^T(\langle Y_0|,\langle Y_0|)\Big)}{\det\big(I_\sigma+\Omega^*\Omega\big)}-1\right], \tag{2.55}$$

where

$$\langle Y_0|=(1,0,\ldots,0)_{1\times N}. \tag{2.56}$$

Proof. From the expansion of $M_{12}(x,t,k)$, it follows that

$$\begin{aligned}q(x,t)&=\lim_{k\to\infty}2ikM_{12}(x,t,k)\\&=\lim_{k\to\infty}2ik\big(\langle Y^*(k^*)|,\langle\tilde{Y}^*(k^*)|\big)\big(|G\rangle,|L\rangle\big)^T\\&=\lim_{k\to\infty}2ik\big(\langle Y^*(k^*)|,\langle\tilde{Y}^*(k^*)|\big)\big(I_\sigma+\Omega^*\Omega\big)^{-1}\big(|\eta\rangle,-|\tilde{\eta}\rangle\big)^T\\&=2i\big(\langle Y_0|,\langle Y_0|\big)\big(I_\sigma+\Omega^*\Omega\big)^{-1}\big(|\eta\rangle,-|\tilde{\eta}\rangle\big)^T\\&=2i\left[\frac{\det\Big(I_\sigma+\Omega^*\Omega+(|\eta\rangle,-|\tilde{\eta}\rangle)^T(\langle Y_0|,\langle Y_0|)\Big)}{\det\big(I_\sigma+\Omega^*\Omega\big)}-1\right].\end{aligned}$$

2.4. Multiple high-order pole solutions

Now we will study the general case that $s_{22}(k)$ has N high-order zero points k_1, k_2,…,k_N, $k_i \in D^+$ for $i = 1, 2, \ldots, N$, and their powers are n_1, n_2,…,n_N, respectively. Let $r_j(k)$ be $r(k)'s$ Laurent series in $k = \pm k_j$, like the case of one high-order pole discussed above, we can obtain

$$r_j(k) = r_{j,0}(k) + \sum_{m_j=1}^{n_j} \frac{r_{j,m_j}}{(k-k_j)^{m_j}}, \qquad r_j^*(k^*) = r_{j,0}^*(k^*) + \sum_{m_j=1}^{n_j} \frac{r_{j,m_j}^*}{(k-k_j^*)^{m_j}}, \tag{2.57}$$

$$r_j(k) = \tilde{r}_{j,0}(k) + \sum_{m_j=1}^{n_j} \frac{(-1)^{m_j+1} r_{j,m_j}}{(k+k_j)^{m_j}}, \qquad r_j^*(k^*) = \tilde{r}_{j,0}^*(k^*) + \sum_{m_j=1}^{n_j} \frac{(-1)^{m_j+1} r_{j,m_j}^*}{(k+k_j^*)^{m_j}}, \tag{2.58}$$

where

$$r_{j,m_j} = \lim_{k\to k_j} \frac{1}{(n_j-m_j)!} \frac{\partial^{n_j-m_j}}{\partial k^{n_j-m_j}} \left[(k-k_j)^{n_j} r(k)\right],$$

and $r_{j,0}(k)$ $(j = 1, \ldots, N)$ is analytic for all $k \in D^+$.

By the similar method in above, the multiple solitons of the GI equation are obtained as follows.

Theorem 2. *With the rapidly decaying initial condition (1.5), if $s_{22}(k)$ has N distinct high-order poles, then the multiple solitons of GI equation have the same form as (2.55)*

$$q(x,t) = 2i\Big[\frac{\det\Big(I_\sigma + \Omega^*\Omega + |\eta\rangle\langle Y_0|\Big)}{\det\left(I_\sigma + \Omega^*\Omega\right)} - 1\Big], \tag{2.59}$$

where

$$|\eta\rangle = [|\eta_1\rangle, |\eta_2\rangle, \ldots, |\eta_N\rangle]^T, \quad |\eta_j\rangle = [\eta_{j,1}, \eta_{j,2}, \ldots, \eta_{j,n_j}, -\tilde{\eta}_{j,1}, -\tilde{\eta}_{j,2}, \ldots, -\tilde{\eta}_{j,n_j}], \tag{2.60a}$$

$$\eta_{j,l} = \sum_{m_j=l}^{n_j} r_{j,m_j} f_{m_j-l}(x,t), \qquad \tilde{\eta}_{j,l} = \sum_{m_j=l}^{n_j} (-1)^{l+1} r_{j,m_j} f_{m_j-l}(x,t), \tag{2.60b}$$

$$\langle Y_0| = [\langle Y_1^0|, \langle Y_2^0|, \ldots, \langle Y_N^0|], \qquad \langle Y_j^0| = [\langle Y_j^{00}|, \langle Y_j^{00}|], \qquad \langle Y_j^{00}| = [1, 0, \ldots, 0]_{1\times n_j}, \tag{2.60c}$$

$$\Omega = \begin{pmatrix} [\omega_{11}] & [\omega_{12}] & \cdots & [\omega_{1N}] \\ [\omega_{21}] & [\omega_{22}] & \cdots & [\omega_{2N}] \\ \vdots & \vdots & \ddots & \vdots \\ [\omega_{N1}] & [\omega_{N2}] & \cdots & [\omega_{NN}] \end{pmatrix}, \qquad [\omega_{jl}]_{2n_j\times 2n_l} = \begin{pmatrix} [w_{jl}^1]_{n_j\times n_l} & [\omega_{jl}^2]_{n_j\times n_l} \\ [w_{jl}^3]_{n_j\times n_l} & [\omega_{jl}^4]_{n_j\times n_l} \end{pmatrix}, \tag{2.60d}$$

$$w_{jl,pq}^1 = -\sum_{m_j=p}^{n_j} \sum_{s_j=0}^{m_j-p} \binom{q+s_j-1}{s_j} \frac{(-1)^{s_j} r_{j,m_j}^* f_{j,m_j-p-s_j}^*}{(k_j^*-k_l)^{s_j+q}}, \tag{2.60e}$$

$$w_{jl,pq}^2 = -\sum_{m_j=p}^{n_j} \sum_{s_j=0}^{m_j-p} \binom{q+s_j-1}{s_j} \frac{(-1)^{s_j} r_{j,m_j}^* f_{j,m_j-p-s_j}^*}{(k_j^*+k_l)^{s_j+q}}, \tag{2.60f}$$

$$w_{jl,pq}^3 = \sum_{m_j=p}^{n_j} \sum_{s_j=0}^{m_j-p} \binom{q+s_j-1}{s_j} \frac{(-1)^{p+s_j+1} r_{j,m_j}^* f_{j,m_j-p-s_j}^*}{(k_j^*-k_l)^{s_j+q}}, \tag{2.60g}$$

$$w_{jl,pq}^4 = \sum_{m_j=p}^{n_j} \sum_{s_j=0}^{m_j-p} \binom{q+s_j-1}{s_j} \frac{(-1)^{p+s_j+1} r_{j,m_j}^* f_{j,m_j-p-s_j}^*}{(k_j^*+k_l)^{s_j+q}}, \tag{2.60h}$$

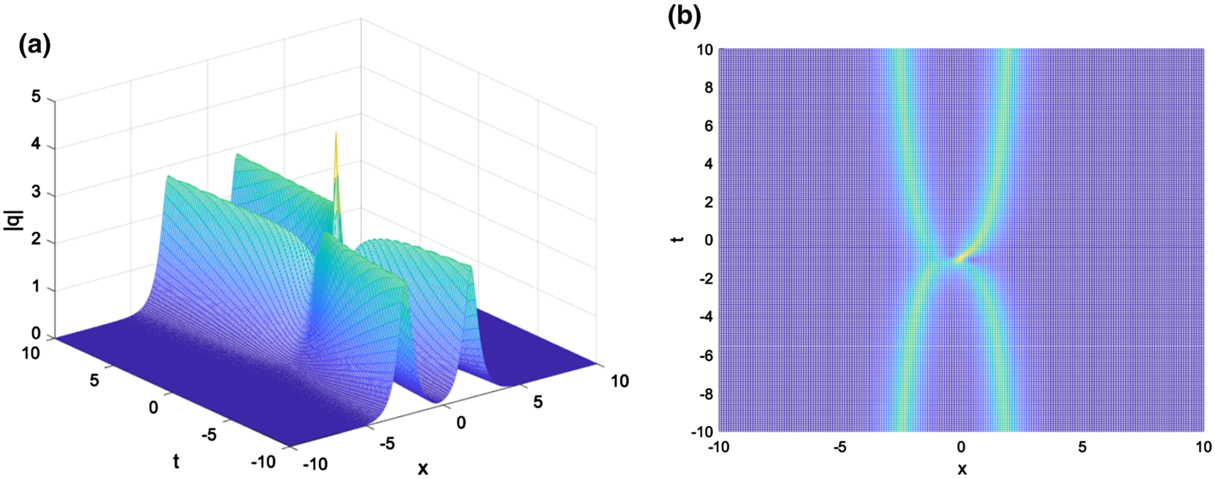

FIG. 2. One-soliton with one second-order pole, here taking parameters $r_1 = 1$, $r_2 = 2$, $k_0 = e^{\frac{\pi i}{4}}$, $k_0^* = e^{-\frac{\pi i}{4}}$. **a** The three-dimensional graph. **b** The contour of the wave

$$I_\sigma = \begin{pmatrix} I_{n_1\times n_1} & & & & \\ & -I_{n_1\times n_1} & & & \\ & & \ddots & & \\ & & & I_{n_N\times n_N} & \\ & & & & -I_{n_N\times n_N} \end{pmatrix}. \tag{2.60i}$$

We then give the figure of one-soliton solution with one second-order pole (Fig. 2).

3. IST with nonzero boundary and high-order poles

3.1. Riemann surface and uniformization variable

To make convenience for the later calculation, we handle the Lax pair (2.1) and the boundary condition (1.6) at the beginning. We make a gauge transformation

$$q \longrightarrow q e^{-\frac{3}{2}iq_0^4 t + iq_0^2 x},$$
$$\phi \longrightarrow e^{(-\frac{3}{4}iq_0^4 t + \frac{1}{2}iq_0^2 x)\sigma_3}\phi.$$

The GI equation (1.4) then becomes

$$iq_t + q_{xx} + 2iq_0^2 q_x - iq^2 q_x^* - q_0^2 q^2 q^* + \frac{1}{2}q^3 q^{*2} + \frac{1}{2}q_0^4 q = 0, \tag{3.1}$$

with corresponding boundary

$$\lim_{x\to\pm\infty} q(x,t) = q_\pm, \tag{3.2}$$

where $|q_\pm| = q_0$.

The GI equation (3.1) is the compatibility condition of the Lax pair

$$\phi_x = X\phi, \qquad \phi_t = T\phi, \tag{3.3}$$

where

$$X=-ik^2\sigma_3+\frac{i}{2}(|q|^2-q_0^2)\sigma_3+kQ,\qquad Q=\begin{pmatrix}0 & q\\ -q^* & 0\end{pmatrix},$$

$$T=-2ik^4\sigma_3+(ik^2|q|^2-iq_0^2|q|^2+\frac{i}{4}|q|^4+\frac{3}{4}iq_0^4)\sigma_3+\frac{1}{2}(Q_xQ-QQ_x)$$
$$+2k^3Q-ikQ_x\sigma_3-kq_0^2Q.$$

Under the boundary (3.2), asymptotic spectral problem of the Lax pair (3.3) becomes

$$\phi_x=X_\pm\phi,\qquad \phi_t=T_\pm\phi, \tag{3.4}$$

where

$$X_\pm=-ik^2\sigma_3+kQ_\pm,\qquad T_\pm=(2k^2-q_0^2)X_\pm, \tag{3.5}$$

and

$$Q_\pm=\begin{pmatrix}0 & q_\pm\\ -q_\pm^* & 0\end{pmatrix}.$$

The eigenvalues of the matrix $X_\pm$ are $\pm ik\lambda$, where $\lambda^2=k^2+q_0^2$. Since the eigenvalues are doubly branched, we introduce the two-sheeted Riemann surface defined by

$$\lambda^2=k^2+q_0^2, \tag{3.6}$$

then $\lambda(k)$ is single-valued on this surface. The branch points are $k=\pm iq_0$. Letting

$$k+iq_0=r_1e^{i\theta_1},\quad k-iq_0=r_2e^{i\theta_2},$$

we can get two single-valued analytic functions on the Riemann surface

$$\lambda(k)=\begin{cases}(r_1r_2)^{1/2}e^{i(\theta_1+\theta_2)/2}, & \text{on } S_1,\\[2mm] -(r_1r_2)^{1/2}e^{i(\theta_1+\theta_2)/2}, & \text{on } S_2,\end{cases} \tag{3.7}$$

where $-\pi/2<\theta_j<3/2\pi$ for $j=1,2$.

Gluing the two copies of the complex plane S_1 and S_2 along the segment $[-iq_0,iq_0]$, we then obtain the Riemann surface. Along the real k axis, we have $\lambda(k)=\pm\mathrm{sign}(k)\sqrt{k^2+q_0^2}$, where the "±" applies on S_1 and S_2 of the Riemann surface, respectively, and where the square root sign denotes the principal branch of the real-valued square root function.

Next, we take a uniformization variable

$$z=k+\lambda, \tag{3.8}$$

then we obtain two single-valued functions

$$k(z)=\frac{1}{2}(z-\frac{q_0^2}{z}),\qquad \lambda(z)=\frac{1}{2}(z+\frac{q_0^2}{z}). \tag{3.9}$$

This implies that we can discuss the scattering problem on a standard z-plane instead of the two-sheeted Riemann surface by the inverse mapping. We define D^+, D^- and Σ on z-plane as

$$\Sigma=\mathbb{R}\cup i\mathbb{R}\backslash\{0\},\qquad D^+=\{z:\mathrm{Re}z\mathrm{Im}z>0\},\qquad D^-=\{z:\mathrm{Re}z\mathrm{Im}z<0\}.$$

the two domains are shown in Fig. 3.

From these discussions, we can derive that

$$\mathrm{Im}(k(z)\lambda(z))=\mathrm{Im}\frac{z^4-q_0^4}{4z^2}=\mathrm{Im}\frac{(|z|^4+q_0^4)z^2-2q_0^4((\mathrm{Re}z)^2-(\mathrm{Im}z)^2)}{4|z|^4}$$
$$=\frac{1}{4|z|^4}(|z|^4+q_0^4)\mathrm{Im}z^2=\frac{1}{2|z|^4}(|z|^4+q_0^4)\mathrm{Re}z\mathrm{Im}z,$$

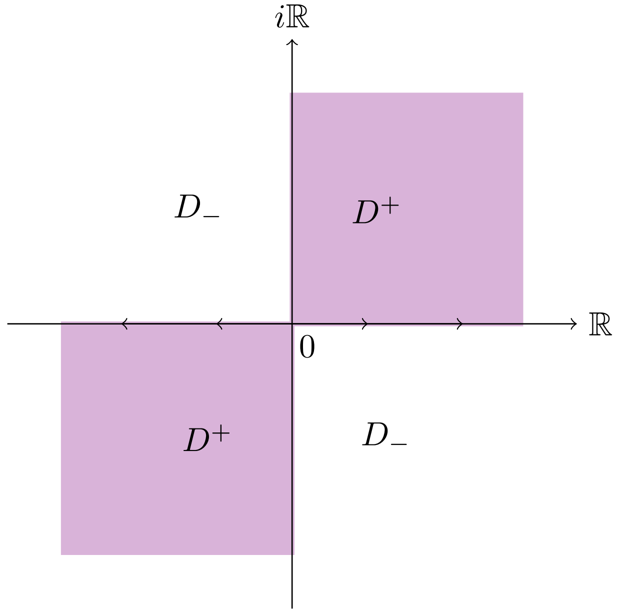

Fig. 3. Complex z-plane consists of the region D^+ (the violet regions) and the D^- (the white regions)

which implies that

$$\operatorname{Im}(k(z)\lambda(z))\begin{cases} = 0, \text{ as } z \in \Sigma \\ > 0, \text{ as } z \in D^+ \\ < 0, \text{ as } z \in D^- \end{cases}. \tag{3.10}$$

3.2. Spectral analysis

3.2.1. Eigenfunctions and scattering matrix. For eigenvalue $\pm i\lambda$, we can write the asymptotic eigenvector matrix as

$$Y_\pm = \begin{pmatrix} 1 & -\frac{iq_\pm}{z} \\ -\frac{iq_\pm^*}{z} & 1 \end{pmatrix} = I - \frac{i}{z}\sigma_3 Q_\pm, \tag{3.11}$$

so that $X_\pm$ and $T_\pm$ can be diagonalized by $Y_\pm$

$$X_\pm = Y_\pm(-ik\lambda\sigma_3)Y_\pm^{-1}, \qquad T_\pm = Y_\pm(-(2k^2 - q_0^2)ik\lambda\sigma_3)Y_\pm^{-1}. \tag{3.12}$$

Direct computation shows that

$$\det(Y_\pm) = 1 + \frac{q_0^2}{z^2} \triangleq \gamma, \tag{3.13}$$

and

$$Y_\pm^{-1} = \frac{1}{\gamma}\begin{pmatrix} 1 & \frac{iq_\pm}{z} \\ \frac{iq_\pm^*}{z} & 1 \end{pmatrix} = \frac{1}{\gamma}(I + \frac{i}{z}\sigma_3 Q_\pm), \qquad z \neq \pm iq_0. \tag{3.14}$$

Substituting (3.12) into (3.4), we immediately obtain

$$(Y_\pm^{-1}\psi)_x = -ik\lambda\sigma_3(Y_\pm^{-1}\psi), \qquad (Y_\pm^{-1}\psi)_t = -(2k^2 - q_0^2)ik\lambda\sigma_3(Y_\pm^{-1}\psi), \qquad z \neq \pm iq_0, \tag{3.15}$$

from which we can derive the solution of the asymptotic spectral problem (3.4)

$$\psi(x,t,z)=\begin{cases} Y_\pm e^{i\theta(z)\sigma_3}, & z\neq \pm iq_0,\\ I+(x-3q_0^2t)Y_\pm(z), & z=\pm iq_0,\end{cases} \tag{3.16}$$

where

$$\theta(x,t,z)=-k(z)\lambda(z)[x+(2k^2(z)-q_0^2)t].$$

For convenience, we will omit x and t dependence in $\theta(x,t,z)$ henceforth.

We define the Jost eigenfunctions $\phi_\pm(x,t,z)$ as the simultaneous solutions of both parts of the Lax pair so that

$$\phi_\pm=Y_\pm e^{i\theta(z)\sigma_3}+o(1),\qquad x\to\pm\infty. \tag{3.17}$$

We introduce modified eigenfunctions by factorizing the asymptotic exponential oscillations

$$\mu_\pm=\phi_\pm e^{-i\theta(z)\sigma_3}, \tag{3.18}$$

then we have

$$\mu_\pm\sim Y_\pm,\qquad x\to\pm\infty.$$

Meanwhile, $\mu_\pm$ acquire the equivalent Lax pair

$$(Y_\pm^{-1}\mu_\pm)_x-ik\lambda[Y_\pm^{-1}\mu_\pm,\sigma_3]=Y_\pm^{-1}\Delta X_\pm\mu_\pm, \tag{3.19}$$

$$(Y_\pm^{-1}\mu_\pm)_t-ik\lambda(2k^2-q_0^2)[Y_\pm^{-1}\mu_\pm,\sigma_3]=Y_\pm^{-1}\Delta T_\pm\mu_\pm, \tag{3.20}$$

where $\Delta X_\pm=X-X_\pm$ and $\Delta T_\pm=T-T_\pm$. These two equations can be written in full derivative form

$$d(e^{-i\theta(z)\hat\sigma_3}Y_\pm^{-1}\mu_\pm)=e^{-i\theta(z)\hat\sigma_3}[Y_\pm^{-1}(\Delta X_\pm \mathrm{d}x+\Delta T_\pm \mathrm{d}t)\mu_\pm], \tag{3.21}$$

which leads to the Volterra integral equations

$$\mu_\pm(x,t,z)=\begin{cases} Y_\pm+\int_{\pm\infty}^x Y_\pm e^{-ik\lambda(x-y)\hat\sigma_3}[Y_\pm^{-1}\Delta X_\pm(y,t)\mu_\pm(y,t,z)]\mathrm{d}y, & z\neq\pm iq_0,\\ Y_\pm+\int_{\pm\infty}^x[I+(x-y)X_\pm(z)]\Delta X_\pm(y,t)\mu_\pm(y,t,z)\mathrm{d}y, & z=\pm iq_0,\end{cases} \tag{3.22}$$

where we define $e^{\alpha\hat\sigma}A:=e^{\alpha\sigma}Ae^{-\alpha\sigma}$, for a matrix A.

Since $\mathrm{tr}X=\mathrm{tr}T=0$ in (3.3), then by using Abel formula, we have

$$(\det\phi_\pm)_x=(\det\phi_\pm)_t=0,\qquad \det(\mu_\pm)=\det(\phi_\pm e^{-i\theta(z)\sigma_3})=\det(\phi_\pm).$$

So that $(\det\mu_\pm)_x=(\det\mu_\pm)_t=0$, which means $\det(\mu_\pm)$ is independent with x,t. Furthermore, we know that $\mu_\pm$ is invertible from

$$\det\mu_\pm=\lim_{x\to\pm\infty}\det(\mu_\pm)=\det Y_\pm=\gamma\neq0,\qquad x,t\in\mathbb{R},\qquad z\in\Sigma_0. \tag{3.23}$$

Since $\phi_\pm$ are two fundamental matrix solutions of the linear Lax pair (3.3), there exists a relation between ϕ_+ and ϕ_-

$$\phi_+(x,t,z)=\phi_-(x,t,z)S(z),\qquad x,t\in\mathbb{R},\qquad z\in\Sigma_0, \tag{3.24}$$

where $S(z)$ is called scattering matrix and (3.23) implies that $\det S(z)=1$. Letting $S(z)=(s_{ij})$, for the individual columns

$$\phi_{+,1}=s_{11}\phi_{-,1}+s_{21}\phi_{-,2},\qquad \phi_{+,2}=s_{12}\phi_{-,1}+s_{22}\phi_{-,2}. \tag{3.25}$$

By using (3.24), we obtain

$$s_{11}(z)=\frac{\mathrm{Wr}(\phi_{+,1},\phi_{-,2})}{\gamma},\qquad s_{12}(z)=\frac{\mathrm{Wr}(\phi_{+,2},\phi_{-,2})}{\gamma}, \tag{3.26}$$

$$s_{21}(z)=\frac{\mathrm{Wr}(\phi_{-,1},\phi_{+,1})}{\gamma},\qquad s_{22}(z)=\frac{\mathrm{Wr}(\phi_{-,1},\phi_{+,2})}{\gamma}. \tag{3.27}$$

3.2.2. Analyticity. Here we directly state the analyticity of eigenfunctions $\mu_\pm$ and scattering data $s_{11},\ s_{22}$, the detail proofs of them were given in our paper [39].

Proposition 5. *Suppose $q(x,t)-q_\pm \in L^1(\mathbb{R}^\pm)$, then the Volterra integral equation (3.22) has unique solutions $\mu_\pm(x,t,z)$ defined by (3.18) in $\Sigma_0 := \Sigma \setminus \{\pm iq_0\}$. Moreover, the columns $\mu_{-,1}$ and $\mu_{+,2}$ can be analytically extended to D^+ and continuously extended to $D^+ \cup \Sigma_0$, while the columns $\mu_{+,1}$ and $\mu_{-,2}$ can be analytically extended to D^- and continuously extended to $D^- \cup \Sigma_0$, where $\mu_{\pm,j}(x,t,z)(j=1,2)$ denote the j-th column of $\mu_\pm$.*

Proposition 6. *Suppose $(1+|x|)(q(x,t)-q_\pm) \in L^1(\mathbb{R}^\pm)$, then the Volterra integral equation (3.22) has unique solutions $\mu_\pm(x,t,z)$ defined by (3.18) in Σ. Besides, the columns $\mu_{-,1}$ and $\mu_{+,2}$ can be analytically extended to D^+ and continuously extended to $D^+ \cup \Sigma$, while the columns $\mu_{+,1}$ and $\mu_{-,2}$ can be analytically extended to D^- and continuously extended to $D^- \cup \Sigma$.*

Lemma 1. *Consider an n-dimensional first-order homogeneous linear ordinary differential equation, $\mathrm{d}y(x)/\mathrm{d}x = A(x)y(x)$, on an interval $\mathbb{D} \in \mathbb{R}$, where $A(x)$ denotes a complex square matrix of order n. Let Φ be a matrix-valued solution of this equation. If the trace $trA(x)$ is a continuous function, then one has*

$$\det \Phi(x) = \det \Phi(x_0) \exp\Big[\int_{x_0}^{x} trA(\xi)\mathrm{d}\xi\Big], \qquad x, x_0 \in \mathbb{D}. \tag{3.28}$$

Proposition 7. *The Jost solutions $\Phi(x,t,z)$ are the simultaneous solutions of both parts of the Lax pair (3.3).*

Proposition 8. *Suppose $q(x,t)-q_\pm \in L^1(\mathbb{R}^\pm)$. Then s_{11} can be analytically extended to D^- and continuously extended to $D^- \cup \Sigma_0$, while s_{22} can be analytically extended to D^+ and continuously extended to $D^+ \cup \Sigma_0$. Moreover, s_{12} and s_{21} are continuous in Σ_0.*

Note that we cannot exclude the possible existence of zeros for $s_{11}(z)$ and $s_{22}(z)$ along Σ_0. To solve the RH problem, we restrict our consideration to potentials without spectral singularities, i.e., $s_{11}(z) \neq 0$, $s_{22}(z) \neq 0$ for $z \in \Sigma$. Besides, we assume that the scattering coefficients are continuous at the branch points. The reflection coefficients which will be needed in the inverse problem are

$$\tilde{r}(z) = \frac{s_{21}}{s_{11}}, \qquad r(z) = \frac{s_{12}}{s_{22}}. \tag{3.29}$$

3.2.3. Symmetries. For the GI equation with nonzero boundary, we not only need to deal with the map $k \mapsto k^*$, but also need to pay attention to the sheets of the Riemann surface. We can see from the Riemann surface that the transformation $z \mapsto z^*$ implies $(k,\lambda) \mapsto (k^*,\lambda^*)$ and $z \mapsto -q_0^2/z$ implies $(x,\lambda) \mapsto (k,-\lambda)$. Therefore, we would like to discuss the symmetries in the following way.

Proposition 9. *The Jost solution, scattering matrix and reflection coefficients satisfy the following reduction conditions on z-plane*

- *The first symmetry reduction*

$$\phi_\pm(x,t,z) = \sigma_2 \phi_\pm^*(x,t,z^*)\sigma_2, \qquad S(z) = \sigma_2 S^*(z^*)\sigma_2, \qquad r(z) = -\tilde{r}^*(z^*), \tag{3.30}$$

where $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

- *The second symmetry reduction*

$$\phi_\pm(x,t,z) = \sigma_1 \phi_\pm^*(x,t,-z^*)\sigma_1, \qquad S(z) = \sigma_1 S^*(-z^*)\sigma_1, \qquad r(z) = \tilde{r}^*(-z^*), \tag{3.31}$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

- *The third symmetry reduction*

$$\phi_\pm(x,t,z) = -\frac{i}{z}\phi_\pm(x,t,-\frac{q_0^2}{z})\sigma_3 Q_\pm, \tag{3.32}$$

$$S(z) = (\sigma_3 Q_-)^{-1} S(-\frac{q_0^2}{z})\sigma_3 Q_+, \qquad r^*(z^*) = \frac{q_-}{q_-^*}\tilde{r}(-\frac{q_0^2}{z}). \tag{3.33}$$

3.2.4. Asymptotic behaviors. To solve the RH problem in the next section, it is necessary to discuss the asymptotic behaviors of the modified Jost solutions and scattering matrix as $z \to \infty$ and $z \to 0$ by the standard Wentzel–Kramers–Brillouin (WKB) expansions.

Proposition 10. *The asymptotic behaviors for the modified Jost solutions are given as*

$$\mu_\pm(x,t,z) = I + o(z^{-1}), \qquad z \to \infty, \tag{3.34}$$

$$\mu_\pm(x,t,z) = -\frac{i}{z}\sigma_3 Q_\pm + o(1), \qquad z \to 0. \tag{3.35}$$

From (3.34), we can get that

$$q(x,t) = \lim_{z\to\infty} iz\mu_\pm^{(12)}, \tag{3.36}$$

which will be used in the following. Inserting the above asymptotic behaviors for the modified Jost eigenfunctions into the Wronskian representation (3.26) and (3.27), with a little calculations, we get the asymptotic behaviors of the scattering matrix.

Proposition 11. *The asymptotic behaviors of the scattering matrix are*

$$S(z) = I + O(z^{-1}), \qquad z \to \infty, \tag{3.37}$$

$$S(z) = \mathrm{diag}(\frac{q_-}{q_+}, \frac{q_+}{q_-}) + O(z), \qquad z \to 0. \tag{3.38}$$

3.2.5. Distribution of spectrum. The discrete spectrum of the scattering problem is the set of all values $z \in \mathbb{C} \setminus \Sigma$, for which eigenfunctions exist in $L^2(\mathbb{R})$. We would like to show that these values are the zeros of $s_{11}(z)$ in D^- and those of $s_{22}(z)$ in D^+.

We can show that the uniformization transformation (3.9) changes the segment $[-iq_0, iq_0]$ on k-plane into the circle $|z| = q_0$ on z-plane. We suppose that s_{22} has one Nth-order zero z_0 in $D^+ \cap \{z \in \mathbb{C} : \mathrm{Im} z > 0, |z| > q_0\}$, then symmetries (3.30)-(3.33) imply that

$$s_{22}(\pm z_0) = 0 \Leftrightarrow s_{11}^*(\pm z_0^*) = 0 \Leftrightarrow s_{11}(\pm\frac{q_0^2}{z_0}) = 0 \Leftrightarrow s_{22}(\pm\frac{q_0^2}{z_0^*}) = 0. \tag{3.39}$$

Therefore, the discrete spectrum is the set

$$Z = \left\{\pm z_0, \pm z_0^*, \pm\frac{q_0^2}{z_0}, \pm\frac{q_0^2}{z_0^*}\right\}, \tag{3.40}$$

which can be seen in Fig. 4.

3.3. Riemann–Hilbert Problem

As we all know, the equation (3.24) is the beginning of the formulation of the inverse problem. We always regard it as a relation between eigenfunctions analytic in D^+ and those analytic in D^-. Thus, it is necessary for us to introduce the following RH problem.

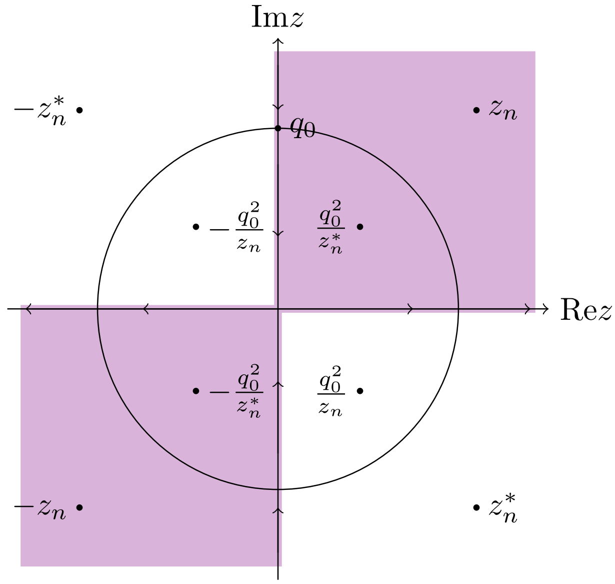

FIG. 4. Distribution of the the discrete spectrum and the contours for the RH problem on complex z-plane

Proposition 12. *Define the sectionally meromorphic matrix*

$$M(x,t,z) = \begin{cases} M^- = \begin{pmatrix} \frac{\mu_{+,1}}{s_{11}} & \mu_{-,2} \end{pmatrix}, \text{ as } z \in D^-, \\ M^+ = \begin{pmatrix} \mu_{-,1} & \frac{\mu_{+,2}}{s_{22}} \end{pmatrix}, \text{ as } z \in D^+. \end{cases} \tag{3.41}$$

Then a multiplicative matrix RH problem is proposed:

- *Analyticity: $M(x,t,z)$ is analytic in $\mathbb{C} \setminus \Sigma$ and has single poles.*
- *Jump condition*

$$M^-(x,t,z) = M^+(x,t,z)(I - G(x,t,z)), \qquad z \in \Sigma, \tag{3.42}$$

where

$$G(x,t,z) = \begin{pmatrix} r(z)\tilde{r}(z) & e^{2i\theta} r(z) \\ -e^{-2i\theta}\tilde{r}(z) & 0 \end{pmatrix}. \tag{3.43}$$

- *Asymptotic behaviors*

$$M(x,t,z) \sim I + O(z^{-1}), \qquad z \to \infty, \tag{3.44}$$

$$M(x,t,z) \sim -\frac{i}{z}\sigma_3 Q_- + O(1), \qquad z \to 0. \tag{3.45}$$

From (3.36), we know that

$$q(x,t) = \lim_{z\to\infty} izM^{(12)}. \tag{3.46}$$

3.4. Single high-order pole solutions

Let $z_0 \in D^+$ be the Nth-order pole, from the symmetries (3.30)-(3.32) it is obvious that $-z_0$, $\pm\frac{q_0^2}{z_0^*} \in D^+$ also is the Nth-order pole of $s_{22}(z)$. Then $\pm z_0^*$ and $\pm\frac{q_0^2}{z_0}$ are the Nth-order poles of $s_{11}(z)$. The discrete spectrum is the set

$$\left\{\pm z_0, \pm z_0^*, \pm\frac{q_0^2}{z_0^*}, \pm\frac{q_0^2}{z_0}\right\}, \tag{3.47}$$

which can be seen in Fig. 4

Let $\nu_1 = z_0$, $\nu_2 = \frac{q_0^2}{z_0^*}$, then the discrete spectrum is $\{\pm\nu_1, \pm\nu_2, \pm\nu_1^*, \pm\nu_2^*\}$. Let

$$s_{22}(z) = (z^2-\nu_1^2)^N(z^2-\nu_2^2)^N s_0(z), \tag{3.48}$$

in which $s_0(z) \neq 0$ in D^+. According to the Laurent series expansion in poles, $r(z)$ and $r^*(z^*)$ can be, respectively, expanded as

$$r_j(z) = r_{0,j}(z) + \sum_{m_j=1}^{N}\frac{r_{j,m_j}}{(z-\nu_j)^{m_j}}, \quad \text{in } z=\nu_j, \quad j=1,2; \tag{3.49a}$$

$$r_j(z) = \tilde{r}_{0,j}(z) + \sum_{m_j=1}^{N}\frac{(-1)^{m_j+1}r_{j,m_j}}{(z+\nu_j)^{m_j}} \quad \text{in } z=-\nu_j, \quad j=1,2; \tag{3.49b}$$

$$r_j^*(z^*) = r_{0,j}^*(z^*) + \sum_{m_j=1}^{N}\frac{r_{j,m_j}^*}{(z-\nu_j^*)^{m_j}} \quad \text{in } z=\nu_j^*, \quad j=1,2; \tag{3.49c}$$

$$r_j^*(z^*) = \tilde{r}_{0,j}^*(z^*) + \sum_{m_j=1}^{N}\frac{(-1)^{m_j+1}r_{j,m_j}^*}{(z+\nu_j^*)} \quad \text{in } z=-\nu_j^*, \quad j=1,2, \tag{3.49d}$$

where r_{j,m_j} are defined by

$$r_{j,m_j} = \lim_{z\to\nu_j}\frac{1}{(N-m_j)!}\frac{\partial^{N-m_j}}{\partial z^{N-m_j}}[(z-\nu_j)^N r_j(z)], \quad m_j = 1,2,\ldots,N. \tag{3.50}$$

and $r_{0,j}(z)$ and $\tilde{r}_{0,j}(z)$ are analytic for all $z \in D^+$. The definition of $M(x,t,k)$ yields that $z=\pm\nu_j$ $(j=1,2)$ are Nth-order poles of M_{12}, while $z=\pm\nu_j^*$ $(j=1,2)$ are Nth-order poles of M_{11}. According to the normalization condition sated in Proposition 13 one can set

$$M_{11}(x,t,z) = 1 + \sum_{j=1}^{2}\sum_{s=1}^{N}\left(\frac{F_{j,s}(x,t)}{(z-\nu_j^*)^s} + \frac{H_{j,s}(x,t)}{(z+\nu_j^*)^s}\right), \tag{3.51a}$$

$$M_{12}(x,t,z) = -\frac{i}{z}q_- + \sum_{j=1}^{2}\sum_{s=1}^{N}\left(\frac{G_{j,s}(x,t)}{(z-\nu_j)^s} + \frac{L_{j,s}(x,t)}{(z+\nu_j)^s}\right), \tag{3.51b}$$

where $F_{j,s}(x,t)$, $H_{j,s}(x,t)$, $G_{j,s}(x,t)$, $L_{j,s}(x,t)(s=1,2,\ldots,N,\ j=1,2)$ are unknown functions which need to be determined. Once these functions are solved, the solution $M(x,t,z)$ of RHP will be obtained and the solutions $q(x,t)$ of the GI equation will be obtained from (3.51).

Now we are in position to solve $F_{j,s}(x,t)$, $H_{j,s}(x,t)$, $G_{j,s}(x,t)$ and $L_{j,s}(x,t)(s=1,2,\ldots,N,\ j=1,2)$. According to Taylor series expansion, one has

$$e^{2i\theta(z)} = \sum_{l=0}^{+\infty}f_{j,l}(x,t)(z-\nu_j)^l, \quad e^{2i\theta(z)} = \sum_{l=0}^{+\infty}(-1)^l f_{j,l}(x,t)(z+\nu_j)^l, \tag{3.52a}$$

$$e^{-2i\theta(z)} = \sum_{l=0}^{+\infty}f_{j,l}^*(x,t)(z-\nu_j^*)^l, \quad e^{-2i\theta(z)} = \sum_{l=0}^{+\infty}(-1)^l f_{j,l}^*(x,t)(z+\nu_j^*)^l, \tag{3.52b}$$

$$M_{11}(x,t,z) = \sum_{l=0}^{+\infty}\mu_{j,l}(x,t)(z-\nu_j)^l, \quad M_{11}(x,t,z) = \sum_{l=0}^{+\infty}(-1)^l\mu_{j,l}(x,t)(z+\nu_j)^l, \tag{3.52c}$$

$$M_{12}(x,t,z)=\sum_{l=0}^{+\infty}\zeta_{j,l}(x,t)(z-\nu_j^*)^l, \qquad M_{12}(x,t,z)=\sum_{l=0}^{+\infty}(-1)^{l+1}\zeta_{j,l}(x,t)(z+\nu_j^*)^l, \tag{3.52d}$$

where

$$f_{j,l}(x,t)=\lim_{z\to\nu_j}\frac{1}{l!}\frac{\partial^l}{\partial z^l}e^{2i\theta(z)}, \tag{3.53a}$$

$$\mu_{j,l}(x,t)=\lim_{z\to\nu_j}\frac{1}{l!}\frac{\partial^l}{\partial z^l}M_{11}(x,t,z), \qquad \zeta_{j,l}(x,t)=\lim_{z\to\nu_j^*}\frac{1}{l!}\frac{\partial^l}{\partial z^l}M_{12}(x,t,z). \tag{3.53b}$$

When $z\in D^+$, we have the expansions in $z=\nu_j$ $(j=1,2)$

$$M_{11}(z)=\mu_{-,11}=\sum_{l=0}^{+\infty}\mu_{j,l}(x,t)(z-\nu_j)^l, \tag{3.54}$$

$$M_{12}(z)=\frac{\mu_{+,22}(x,t,z)}{s_{22}(z)}=e^{2i\theta}r(z)\mu_{-,11}(x,t,z)+\mu_{-,12}(x,t,z) \tag{3.55}$$

comparing the coefficients of $(z-\nu_j)^{-s}$ with (3.51b), we can get

$$G_{j,s}(x,t)=\sum_{m_j=s}^{N}\sum_{l=0}^{m_j-s}r_{j,m_j}f_{j,m_j-s-l}(x,t)\mu_{j,l}(x,t). \tag{3.56}$$

Similarly, from the expansions in $z=-\nu_j$ $(j=1,2)$, we can get that

$$L_{j,s}(x,t)=\sum_{m_j=s}^{N}\sum_{l=0}^{m_j-s}(-1)^{s+1}r_{j,m_j}f_{j,m_j-s-l}(x,t)\mu_{j,l}(x,t). \tag{3.57}$$

By the same method, when $z\in D^-$, we can obtain that

$$F_{j,s}(x,t)=-\sum_{m_j=s}^{N}\sum_{l=0}^{m_j-s}r_{j,m_j}^*f_{j,m_j-s-l}^*(x,t)\zeta_{j,l}(x,t), \tag{3.58}$$

$$H_{j,s}(x,t)=\sum_{m_j=s}^{N}\sum_{l=0}^{m_j-s}(-1)^{s+1}r_{j,m_j}^*f_{j,m_j-s-l}^*(x,t)\zeta_{j,l}(x,t). \tag{3.59}$$

Actually, $\mu_{j,l}(x,t)$ and $\zeta_{j,l}(x,t)$ $(j=1,2)$ can also be expressed by $F_{j,s}(x,t)$, $H_{j,s}(x,t)$, $G_{j,s}(x,t)$ and $L_{j,s}(x,t)$ $(j=1,2)$. Recalling the definitions of $\zeta_{j,l}(x,t)$ and $\mu_{j,l}(x,t)$ $(j=1,2)$ given by (3.53b) and substituting (3.51) into them, we can obtain

$$\zeta_{j,l}(x,t)=\frac{(-1)^{l+1}}{(\nu_j^*)^{l+1}}iq_-+\sum_{p=1}^{2}\sum_{s=1}^{N}\binom{s+l-1}{l}\left\{\frac{(-1)^lG_{p,s}(x,t)}{(\nu_j^*-\nu_p)^{l+s}}+\frac{(-1)^lL_{p,s}(x,t)}{(\nu_j^*+\nu_p)^{l+s}}\right\}, l=0,1,\ldots, \tag{3.60a}$$

$$\mu_{j,l}(x,t)=\begin{cases}1+\sum_{p=1}^{2}\sum_{s=1}^{N}\left\{\frac{F_{p,s}(x,t)}{(\nu_j-\nu_p^*)^s}+\frac{H_{p,s}(x,t)}{(\nu_j+\nu_p^*)^s}\right\}, & l=0;\\ \sum_{p=1}^{2}\sum_{s=1}^{N}\binom{s+l-1}{l}\left\{\frac{(-1)^lF_{p,s}(x,t)}{(\nu_j-\nu_p^*)^{s+l}}+\frac{(-1)^lH_{p,s}(x,t)}{(\nu_j+\nu_p^*)^{s+l}}\right\}, & l=1,2,3,\ldots\end{cases} \tag{3.60b}$$

Using (3.56)–(3.60), we obtain the system

$$\begin{aligned}F_{j,s}(x,t)&=-iq_-\sum_{m_j=s}^{N}\sum_{l=0}^{m_j-s}\frac{(-1)^{l+1}}{(\nu_j^*)^{l+1}}r_{j,m_j}^*f_{j,m_j-s-l}^*(x,t)\\&\quad-\sum_{m_j=s}^{N}\sum_{l=0}^{m_j-s}\sum_{p=1}^{2}\sum_{q=1}^{N}\binom{q+l-1}{l}r_{j,m_j}^*f_{j,m_j-s-l}^*\Big\{\frac{(-1)^lG_{p,q}(x,t)}{(\nu_j^*-\nu_p)^{l+q}}+\frac{(-1)^lL_{p,q}(x,t)}{(\nu_j^*+\nu_p)^{l+q}}\Big\},\end{aligned}\tag{3.61a}$$

$$\begin{aligned}H_{j,s}(x,t)&=iq_-\sum_{m_j=s}^{N}\sum_{l=0}^{m_j-s}\frac{(-1)^{s+l}}{(\nu_j^*)^{l+1}}r_{j,m_j}^*f_{j,m_j-s-l}^*(x,t)\\&\quad+\sum_{m_j=s}^{N}\sum_{l=0}^{m_j-s}\sum_{p=1}^{2}\sum_{q=1}^{N}(-1)^{s+1}\binom{q+l-1}{l}r_{j,m_j}^*f_{j,m_j-s-l}^*\Big\{\frac{(-1)^lG_{p,q}(x,t)}{(\nu_j^*-\nu_p)^{l+q}}+\frac{(-1)^lL_{p,q}(x,t)}{(\nu_j^*+\nu_p)^{l+q}}\Big\},\end{aligned}\tag{3.61b}$$

$$\begin{aligned}G_{j,s}(x,t)&=\sum_{m_j=s}^{N}r_{j,m_j}f_{j,m_j-s}(x,t)\\&\quad+\sum_{m_j=s}^{N}\sum_{l=0}^{m_j-s}\sum_{p=1}^{2}\sum_{q=1}^{N}\binom{q+l-1}{l}r_{j,m_j}f_{j,m_j-s-l}\Big\{\frac{(-1)^lF_{p,q}(x,t)}{(\nu_j-\nu_p^*)^{l+q}}+\frac{(-1)^lH_{p,q}(x,t)}{(\nu_j+\nu_p^*)^{l+q}}\Big\},\end{aligned}\tag{3.61c}$$

$$\begin{aligned}L_{j,s}(x,t)&=\sum_{m_j=s}^{N}(-1)^{s+1}r_{j,m_j}f_{j,m_j-s}(x,t)\\&\quad+\sum_{m_j=s}^{N}\sum_{l=0}^{m_j-s}\sum_{p=1}^{2}\sum_{q=1}^{N}(-1)^{s+1}\binom{q+l-1}{l}r_{j,m_j}f_{j,m_j-s-l}\Big\{\frac{(-1)^lF_{p,q}(x,t)}{(\nu_j-\nu_p^*)^{l+q}}+\frac{(-1)^lH_{p,q}(x,t)}{(\nu_j+\nu_p^*)^{l+q}}\Big\},\end{aligned}\tag{3.61d}$$

Let us define

$$|\eta_j\rangle=(\eta_{j1},\ldots,\eta_{jN})^T,\qquad \eta_{js}=-iq_-\sum_{m_j=s}^{N}\sum_{l=0}^{m_j-s}\frac{(-1)^{l+1}}{(\nu_j^*)^{l+1}}r_{j,m_j}^*f_{j,m_j-s-l}^*(x,t),\qquad j=1,2;$$

$$|\tilde\eta_j\rangle=(\tilde\eta_{j1},\ldots,\tilde\eta_{jN})^T,\qquad \tilde\eta_{js}=iq_-\sum_{m_j=s}^{N}\sum_{l=0}^{m_j-s}\frac{(-1)^{s+l}}{(\nu_j^*)^{l+1}}r_{j,m_j}^*f_{j,m_j-s-l}^*(x,t),\qquad j=1,2;$$

$$|\xi_j\rangle=(\xi_{j1},\ldots,\xi_{jN})^T,\qquad \xi_{js}=\sum_{m_j=s}^{N}r_{j,m_j}f_{j,m_j-s}(x,t),\qquad j=1,2;$$

$$|\tilde\xi_j\rangle=(\tilde\xi_{j1},\ldots,\tilde\xi_{jN})^T,\qquad \tilde\xi_{js}=\sum_{m_j=s}^{N}(-1)^sr_{j,m_j}f_{j,m_j-s}(x,t),\qquad j=1,2;$$

$$\Omega_{jp}=[\Omega_{jp}]_{sq}=-\sum_{m_j=s}^{N}\sum_{l=0}^{m_j-s}\binom{q+l-1}{l}\frac{(-1)^lr_{j,m_j}^*f_{j,m_j-s-l}^*}{(\nu_j^*-\nu_p)^{l+q}},\qquad j,p=1,2;$$

$$\Omega_{j,p+2}=[\Omega_{j,p+2}]_{sq}=-\sum_{m_j=s}^{N}\sum_{l=0}^{m_j-s}\binom{q+l-1}{l}\frac{(-1)^lr_{j,m_j}^*f_{j,m_j-s-l}^*}{(\nu_j^*+\nu_p)^{l+q}},\qquad j,p=1,2,;$$

$$\Omega_{j+2,p} = [\Omega_{j+2,p}]_{sq} = \sum_{m_j=s}^{N} \sum_{l=0}^{m_j-s} (-1)^{s+1} \begin{pmatrix} q+l-1 \\ l \end{pmatrix} \frac{(-1)^l r^*_{j,m_j} f^*_{j,m_j-s-l}}{(\nu_j^* - \nu_p)^{l+q}}, \qquad j,p = 1,2;$$

$$\Omega_{j+2,p+2} = [\Omega_{j+2,p+2}]_{sq} = \sum_{m_j=s}^{N} \sum_{l=0}^{m_j-s} (-1)^{s+1} \begin{pmatrix} q+l-1 \\ l \end{pmatrix} \frac{(-1)^l r^*_{j,m_j} f^*_{j,m_j-s-l}}{(\nu_j^* + \nu_p)^{l+q}}, \qquad j,p = 1,2.$$

$$|F_p\rangle = (F_{p,1}, \ldots, F_{p,N})^T, \qquad |H_p\rangle = (H_{p,1}, \ldots, H_{p,N})^T, \qquad p = 1,2;$$
$$|G_p\rangle = (G_{p,1}, \ldots, G_{p,N})^T, \qquad |L_p\rangle = (L_{p,1}, \ldots, L_{p,N})^T, \qquad p = 1,2.$$

Let

$$\Omega = \begin{pmatrix} \Omega_{11} & \cdots & \Omega_{14} \\ \vdots & \ddots & \vdots \\ \Omega_{41} & \cdots & \Omega_{44} \end{pmatrix}; \tag{3.62}$$

and

$$|\alpha_1\rangle = (|\eta_1\rangle, |\eta_2\rangle, |\tilde{\eta}_1\rangle, |\tilde{\eta}_2\rangle)^T, \qquad |\alpha_2\rangle = (|\xi_1\rangle, |\xi_2\rangle, |\tilde{\xi}_1\rangle, |\tilde{\xi}_2\rangle)^T;$$
$$|K_1\rangle = (|F_1\rangle, |F_2\rangle, |H_1\rangle, |H_2\rangle)^T, \qquad |K_2\rangle = (|G_1\rangle, |G_2\rangle, |L_1\rangle, |L_2\rangle)^T.$$

Using the similar method with zero boundary condition, we have

$$|K_2\rangle = -\Omega^*(I_\sigma + \Omega^*\Omega)^{-1}\alpha_1 + (I_\sigma + \Omega^*\Omega)^{-1}\alpha_2, \tag{3.63}$$

where

$$I_\sigma = \begin{pmatrix} I & & & \\ & I & & \\ & & -I & \\ & & & -I \end{pmatrix}_{4N\times 4N},$$

so that

$$\begin{aligned} M_{12}(x,t,z) &= -\frac{i}{z}q_- + \langle Y|K_2\rangle \\ &= -\frac{i}{z}q_- + \langle Y|(-\Omega^*(I_\sigma + \Omega^*\Omega)^{-1}\alpha_1 + (I_\sigma + \Omega^*\Omega)^{-1}\alpha_2) \\ &= -\frac{i}{z}q_- + \frac{\det(I_\sigma + \Omega^*\Omega + |\alpha_2\rangle\langle Y|) - \det(I_\sigma + \Omega^*\Omega + |\alpha_1\rangle\langle Y|\Omega^*)}{\det(I_\sigma + \Omega^*\Omega)}, \end{aligned} \tag{3.64}$$

where

$$\langle Y| = \Big(\frac{1}{z-\nu_1}, \ldots, \frac{1}{(z-\nu_1)^N}, \frac{1}{z-\nu_2}, \ldots, \frac{1}{(z-\nu_2)^N}, \frac{1}{z+\nu_1}, \ldots, \frac{1}{(z+\nu_1)^N}, \frac{1}{z+\nu_2}, \ldots, \frac{1}{(z+\nu_2)^N}\Big)_{1\times 4N}.$$

Theorem 3. *With the nonzero boundary condition (1.6), the Nth-order soliton of GI equation is*

$$q(x,t) = q_- + i\Big[\frac{\det(I_\sigma + \Omega^*\Omega + |\alpha_2\rangle\langle Y_0|) - \det(I_\sigma + \Omega^*\Omega + |\alpha_1\rangle\langle Y_0|\Omega^*)}{\det(I_\sigma + \Omega^*\Omega)}\Big],, \tag{3.65}$$

where

$$\langle Y_0| = (1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0)_{1\times 4N}. \tag{3.66}$$

3.5. Multiple high-order pole solutions

Now we will study the general case that $s_{22}(z)$ has N high-order zero points $z_1,\ z_2,\dots,z_N,\ z_k\in D^+$ for $k=1,2,\dots,N$, and their powers are $n_1,\ n_2,\dots,n_N$, respectively. Let $\nu_1^k=z_k,\ \nu_2^k=\frac{q_0^2}{z_k^*}$. Let $r_j^k(z)$ be $r(z)'s$ Laurent series in $z=\nu_j^k$ $(j=1,2)$, like the case of one high-order pole discussed above, we can obtain

$$r_j^k(z)=r_{j,0}^k(z)+\sum_{m_j=1}^{n_k}\frac{r_{j,m_j}^k}{(z-\nu_j^k)^{m_j}},\qquad r_j^{*k}(z^*)=r_{j,0}^*(z^*)+\sum_{m_j=1}^{n_k}\frac{r_{j,m_j}^{k*}}{(z-\nu_j^{k*})^{m_j}},\tag{3.67}$$

$$r_j^k(z)=\tilde r_{j,0}(z)+\sum_{m_j=1}^{n_k}\frac{(-1)^{m_j+1}r_{j,m_j}^k}{(z+\nu_j^k)^{m_j}},\qquad r_j^{*k}(z^*)=\tilde r_{j,0}^*(z^*)+\sum_{m_j=1}^{n_k}\frac{(-1)^{m_j+1}r_{j,m_j}^{k*}}{(z+\nu_j^{k*})^{m_j}},\tag{3.68}$$

where

$$r_{j,m_j}^k=\lim_{z\to\nu_j^k}\frac{1}{(n_k-m_j)!}\frac{\partial^{n_k-m_j}}{\partial k^{n_k-m_j}}\left[(z-\nu_j^k)^{n_k}r(z)\right],$$

and $r_{j,0}^k(z)$ $(k=1,\dots,N)$ is analytic for all $z\in D^+$.

By the similar method in above, the multiple solitons of the GI equation are obtained as follows.

Theorem 4. *With the nonzero boundary condition (1.6), if $s_{22}(z)$ has N distinct high-order poles, then the multiple solitons of GI equation have the same form as (3.65)*

$$q(x,t)=q_-+i\Big[\frac{\det(I_\sigma+\Omega^*\Omega+|\alpha_2\rangle\langle Y_0|)-\det(I_\sigma+\Omega^*\Omega+|\alpha_1\rangle\langle Y_0|\Omega^*)}{\det(I_\sigma+\Omega^*\Omega)}\Big],\tag{3.69}$$

where

$$|\alpha_1\rangle=\big(|\alpha_1^1\rangle,\dots,|\alpha_1^N\rangle\big)^T,\qquad |\alpha_1^k\rangle=\big(|\eta_1^k\rangle,|\eta_2^k\rangle,|\tilde\eta_1^k\rangle,|\tilde\eta_2^k\rangle\big)^T,\ k=1,\dots,N,\tag{3.70a}$$

$$|\alpha_2\rangle=\big(|\alpha_2^1\rangle,\dots,|\alpha_2^N\rangle\big)^T,\qquad |\alpha_2^k\rangle=\big(|\xi_2^k\rangle,|\xi_2^k\rangle,|\tilde\xi_1^k\rangle,|\tilde\xi_2^k\rangle\big)^T,\ k=1,\dots,N,\tag{3.70b}$$

$$|\eta_j^k\rangle=[|\eta_{j1}^k\rangle,\dots,|\eta_{jN}^k\rangle]^T,\qquad |\xi_j^k\rangle=[|\xi_{j1}^k\rangle,\dots,|\xi_{jN}^k\rangle]^T,\ j=1,2,\tag{3.70c}$$

$$\eta_{js}^k=-iq_-\sum_{m_j=s}^{n_k}\sum_{l=0}^{m_j-s}\frac{(-1)^{l+1}}{(\nu_j^{k*})^{l+1}}r_{j,m_j}^{k*}f_{j,m_j-s-l}^{k*}(x,t),\qquad j=1,2;\tag{3.70d}$$

$$\tilde\eta_{js}^k=iq_-\sum_{m_j=s}^{n_k}\sum_{l=0}^{m_j-s}\frac{(-1)^{s+l}}{(\nu_j^{k*})^{l+1}}r_{j,m_j}^{k*}f_{j,m_j-s-l}^{k*}(x,t),\qquad j=1,2;\tag{3.70e}$$

$$\xi_{js}^k=\sum_{m_j=s}^{n_k}r_{j,m_j}^kf_{j,m_j-s}^k(x,t),\qquad \tilde\xi_{js}^k=-\sum_{m_j=s}^{n_k}(-1)^{s+1}r_{j,m_j}^kf_{j,m_j-s}^k(x,t),\quad j=1,2;\tag{3.70f}$$

$$\langle Y_0|=[\langle Y_1^0|,\langle Y_2^0|,\dots,\langle Y_N^0|],\qquad \langle Y_k^0|=[1,0,\dots,0,1,0,\dots,0,1,0,\dots,0,1,0,\dots,0]_{1\times4N},\tag{3.70g}$$

$$\Omega=\begin{pmatrix}[\omega_{11}]&[\omega_{12}]&\cdots&[\omega_{1N}]\\ [\omega_{21}]&[\omega_{22}]&\cdots&[\omega_{2N}]\\ \vdots&\vdots&\ddots&\vdots\\ [\omega_{N1}]&[\omega_{N2}]&\cdots&[\omega_{NN}]\end{pmatrix},\qquad [\omega_{kh}]_{4n_k\times4n_h}=\begin{pmatrix}\omega_{kh}^{11}&\cdots&\omega_{kh}^{14}\\ \vdots&\ddots&\vdots\\ \omega_{kh}^{41}&\cdots&\omega_{kh}^{44}\end{pmatrix},\tag{3.70h}$$

$$\omega_{kh}^{jp}=[\omega_{kh}^{jp}]_{sq}=-\sum_{m_j=s}^{n_k}\sum_{l=0}^{m_j-s}\binom{q+l-1}{l}\frac{(-1)^lr_{j,m_j}^{k*}f_{j,m_j-s-l}^{k*}}{(\nu_j^{k*}-\nu_p^k)^{l+q}},\qquad j,p=1,2;\tag{3.70i}$$

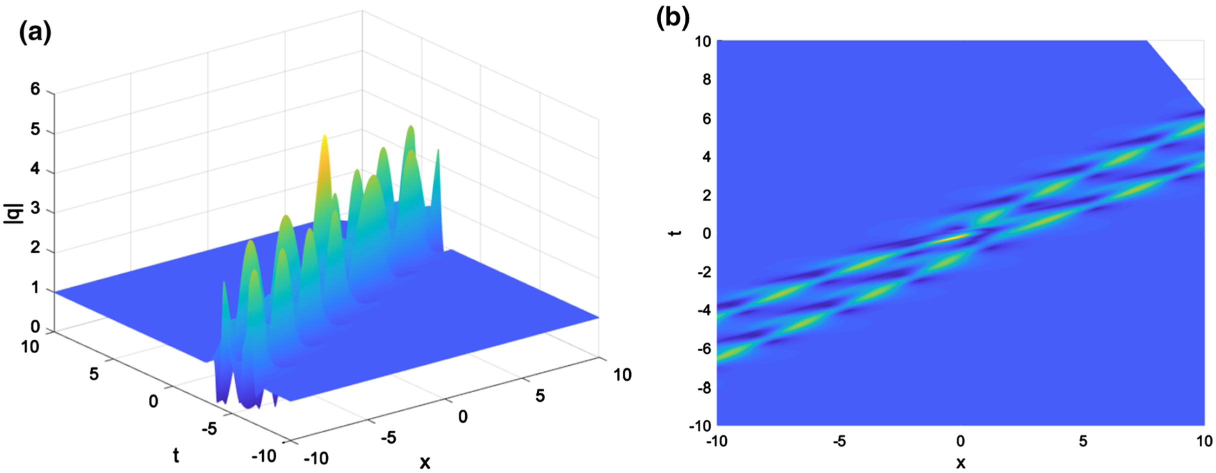

Fig. 5. One-soliton with one second-order pole, here taking parameters $r_{11}=1$, $r_{12}=2$,$r_{21}=4$, $r_{22}=3$, $q_0=1$, $z_0=2e^{\frac{\pi i}{4}}$. **a** The three-dimensional graph. **b** The contour of the wave

$$\omega_{kh}^{j,p+2}=[\omega_{kh}^{j,p+2}]_{sq}=-\sum_{m_j=s}^{n_k}\sum_{l=0}^{m_j-s}\binom{q+l-1}{l}\frac{(-1)^l r_{j,m_j}^{k*}f_{j,m_j-s-l}^{k*}}{(\nu_j^{k*}+\nu_p^k)^{l+q}},\qquad j,p=1,2; \tag{3.70j}$$

$$\omega_{kh}^{j+2,p}=[\omega_{j+2,p}]_{sq}=\sum_{m_j=s}^{n_k}\sum_{l=0}^{m_j-s}(-1)^{s+1}\binom{q+l-1}{l}\frac{(-1)^l r_{j,m_j}^{k*}f_{j,m_j-s-l}^{k*}}{(\nu_j^{k*}-\nu_p^k)^{l+q}},\qquad j,p=1,2; \tag{3.70k}$$

$$\omega_{kh}^{j+2,p+2}=[\omega_{j+2,p+2}]_{sq}=\sum_{m_j=s}^{n_k}\sum_{l=0}^{m_j-s}(-1)^{s+1}\binom{q+l-1}{l}\frac{(-1)^l r_{j,m_j}^{k*}f_{j,m_j-s-l}^{k*}}{(\nu_j^{k*}+\nu_p^k)^{l+q}},\qquad j,p=1,2; \tag{3.70l}$$

$$I_\sigma=\begin{pmatrix} I_{\sigma_1} & & \\ & \vdots & \\ & & I_{\sigma_N}\end{pmatrix},\qquad I_{\sigma_k}=\begin{pmatrix} I & & & \\ & I & & \\ & & -I & \\ & & & -I\end{pmatrix}_{4n_k\times 4n_k},\quad k=1,\ldots,N. \tag{3.70m}$$

We then give the figures of one-soliton solution with one second-order pole (Fig. 5).

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