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Inverse scattering transform for the Gerdjikov–Ivanov equation with nonzero boundary conditions

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Abstract. In this article, we focus on the inverse scattering transform for the Gerdjikov–Ivanov equation with nonzero boundary at infinity. An appropriate two-sheeted Riemann surface is introduced to map the original spectral parameter k into a single-valued parameter z. Based on the Lax pair of the Gerdjikov–Ivanov equation, we derive its Jost solutions with nonzero boundary. Further asymptotic behaviors, analyticity and the symmetries of the Jost solutions and the spectral matrix are in detail derived. The formula of N-soliton solutions is obtained via transforming the problem of nonzero boundary into the corresponding matrix Riemann–Hilbert problem. As examples of N-soliton formula, for $N=1$ and $N=2$, respectively, different kinds of soliton solutions and breather solutions are explicitly presented according to different distributions of the spectrum. The dynamical features of those solutions are characterized in the particular case with a quartet of discrete eigenvalues. It is shown that distribution of the spectrum and non-vanishing boundary also affect feature of soliton solutions.

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Keywords. Gerdjikov–Ivanov equation, Nonzero boundary condition, Riemann–Hilbert problem, Breather solution, Soliton solution.

1. Introduction

It is well known that the nonlinear Schrödinger (NLS) equation [1,2]

$$iq_t + q_{xx} + 2|q|^2 q = 0, \tag{1.1}$$

which is one of the most important integrable systems in the soliton theory, plays an important role and has applications in a wide variety of fields. Besides the NLS equation (1.1), derivative NLS (DNLS) equations were also introduced to investigate the effects of high-order perturbations [3–5]. Among them, there are three deformations of the derivative NLS equations [5]: The first one is Kaup–Newell equation (also called DNLSI equation) [6]

$$iq_t + q_{xx} + i(|q|^2)_x = 0. \tag{1.2}$$

The second type is the Chen–Lee–Liu equation (also called DNLSII equation) [7]

$$iq_t + q_{xx} + i|q|^2 q_x = 0. \tag{1.3}$$

The third type is the Gerdjikov–Ivanov (GI) equation (also called DNLSIII equation) which takes the form [8]

$$iq_t + q_{xx} - iq^2 q_x^* + \frac{1}{2} q^3 q^{*2} = 0, \tag{1.4}$$

where the asterisk $*$ means the complex conjugation. It has been discovered that the three kinds of DNLS equations can be transformed into each another via the gauge transformations [3,4]. The DNLS equations are regarded as models in a wide variety of fields such as weakly nonlinear dispersive water waves, nonlinear optical fibers, quantum field theory and plasmas [9–12]. In plasma physics, the GI equation (1.4) is a model for Alfvén waves propagating parallel to the ambient magnetic field, with q being the transverse magnetic field perturbation and x and t being space and time coordinates, respectively [13,14].

Though there are gauge transformations among the three DNLS equations, such relations turn out to be rather complicated and implicit, which is difficult to apply them to study initial-value problem with nonzero boundary. In this paper, we present a direct approach to the initial-value problem of GI equation via inverse scattering transformation. The GI equation has been studied through many methods, for instance the Darboux transformation [15], the nonlinearization [16,17], the similarity reduction, the bifurcation theory, and others [18,19]. In particular, Riemann–Hilbert method is used to construct N-soliton of the GI equation with zero boundary [20]. However, to our knowledge, there is still no work on GI equation (1.4) with nonzero boundary by using inverse scattering transform or Riemann–Hilbert approach. In this article, we investigate the soliton solution of GI equation (1.4) with the following nonzero boundary:

$$q(x,t) \sim q_{\pm}\mathrm{e}^{-\frac{3}{2}iq_0^4t+iq_0^2x}, \qquad x \to \pm\infty, \tag{1.5}$$

where $|q_{\pm}| = q_0 > 0$, and $q_{\pm}$ are independent of x, t.

It must be admitted that the inverse scattering transform method plays a significant role during the discovery process of the exact solutions of completely integrable systems [21,22]. As a new version of inverse scattering transform method, the Riemann–Hilbert approach has become the preferred research technique to the researchers in investigating the soliton solutions and the long-time asymptotics of integrable systems in recent years [23,24]. More recently, the Riemann–Hilbert approach has become a hot spot to investigate the integrable systems with nonzero boundary [25–29].

This paper is organized as follows: In Sect. 2, we get down to the spectral analysis by introducing an appropriate transformation. Then, we introduce the two-sheeted Riemann surface for uniformization of spectral parameter. In Sects. 3–5, we investigate the Jost solution and scattering matrix and obtain the symmetries of the Jost solution, scattering matrix and reflection coefficients. In Sect. 6, we discuss the discrete spectrum and the residue conditions which are helpful to solve the Riemann–Hilbert problem below. In Sect. 7, we derive the asymptotic behaviors of the Jost solutions and the scattering matrix. In Sect. 8, we search for the connection between the Riemann–Hilbert problem and solution of the GI equation. As a result, the reconstruction formula of the GI equation is expressed by the solution of the RHP. We obtain the trace formula as well as theta condition that reflection coefficients and discrete spectrum satisfy. In Sect. 9, under reflectionless condition, we provide a formula for the N-soliton solutions of GI equation with nonzero boundary. As examples of N-soliton formula, for $N = 1$ and $N = 2$, different kinds of soliton solutions and breather solutions are explicitly presented, respectively, according to different distributions of the spectrum.

2. Spectral analysis

It is well known that the GI equation (1.4) admits the Lax pair [15]

$$\phi_x = X\phi, \qquad \phi_t = T\phi, \tag{2.1}$$

where

$$X = -ik^2\sigma_3 + kQ - \frac{i}{2}Q^2\sigma_3, \tag{2.2}$$

$$T = -2ik^4\sigma_3 + 2k^3Q - ik^2Q^2\sigma_3 - ikQ_x\sigma_3 + \frac{1}{2}(Q_xQ - QQ_x) + \frac{i}{4}Q^4\sigma_3, \tag{2.3}$$

and

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}. \tag{2.4}$$

To make convenience for the later calculation, we handle the Lax pair (2.1) and the boundary condition (1.5) at the beginning. We make a proper transformation

$$q \to q\mathrm{e}^{-\frac{3}{2}iq_0^4t+iq_0^2x},$$
$$\phi \to \mathrm{e}^{(-\frac{3}{4}iq_0^4t+\frac{1}{2}iq_0^2x)\sigma_3}\phi.$$

The GI equation (1.4) then becomes

$$iq_t + q_{xx} + 2iq_0^2q_x - iq^2q_x^* - q_0^2q^2q^* + \frac{1}{2}q^3q^{*2} + \frac{1}{2}q_0^4q = 0, \tag{2.5}$$

with corresponding boundary

$$\lim_{x\to\pm\infty} q(x,t) = q_\pm, \tag{2.6}$$

where $|q_\pm| = q_0$.

The GI equation (2.5) is the compatibility condition of the Lax pair

$$\phi_x = X\phi, \qquad \phi_t = T\phi, \tag{2.7}$$

where

$$X = -ik^2\sigma_3 + \frac{i}{2}(|q|^2 - q_0^2)\sigma_3 + kQ, \qquad Q = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix},$$
$$T = -2ik^4\sigma_3 + (ik^2|q|^2 - iq_0^2|q|^2 + \frac{i}{4}|q|^4 + \frac{3}{4}iq_0^4)\sigma_3 + \frac{1}{2}(Q_xQ - QQ_x)$$
$$+ 2k^3Q - ikQ_x\sigma_3 - kq_0^2Q.$$

Under the boundary (2.6), asymptotic spectral problem of the Lax pair (2.7) becomes

$$\phi_x = X_\pm\phi, \qquad \phi_t = T_\pm\phi, \tag{2.8}$$

where

$$X_\pm = -ik^2\sigma_3 + kQ_\pm, \qquad T_\pm = (2k^2 - q_0^2)X_\pm, \tag{2.9}$$

and

$$Q_\pm = \begin{pmatrix} 0 & q_\pm \\ -q_\pm^* & 0 \end{pmatrix}.$$

The eigenvalues of the matrix $X_\pm$ are $\pm ik\lambda$, where $\lambda^2 = k^2 + q_0^2$. Since the eigenvalues are doubly branched, we introduce the two-sheeted Riemann surface defined by

$$\lambda^2 = k^2 + q_0^2; \tag{2.10}$$

then, $\lambda(k)$ is single-valued on this surface. The branch points are $k = \pm iq_0$. Letting

$$k + iq_0 = r_1\mathrm{e}^{i\theta_1}, \quad k - iq_0 = r_2\mathrm{e}^{i\theta_2},$$

we can get two single-valued analytic functions on the Riemann surface

$$\lambda(k) = \begin{cases} (r_1r_2)^{1/2}\mathrm{e}^{i(\theta_1+\theta_2)/2}, & \text{on } S_1, \\ -(r_1r_2)^{1/2}\mathrm{e}^{i(\theta_1+\theta_2)/2}, & \text{on } S_2, \end{cases} \tag{2.11}$$

where $-\pi/2 < \theta_j < 3/2\pi$ for $j = 1, 2$.

Gluing the two copies of the complex plane S_1 and S_2 along the segment $[-iq_0, iq_0]$, we then obtain the Riemann surface. Along the real k axis we have $\lambda(k) = \pm\,\mathrm{sign}(k)\sqrt{k^2 + q_0^2}$, where the "±" applies on S_1 and S_2 of the Riemann surface, respectively, and where the square root sign denotes the principal branch of the real-valued square root function.

Next, we take a uniformization variable

$$z = k + \lambda; \tag{2.12}$$

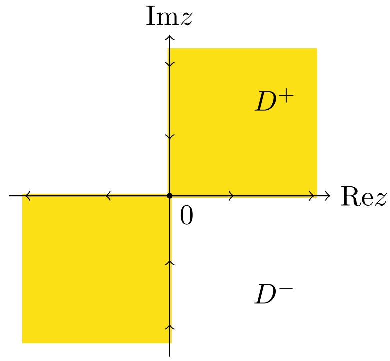

FIG. 1. Complex z-plane consist of the region D^+ (the yellow regions) and the D^- (the white regions)

then, we obtain two single-valued functions

$$k(z) = \frac{1}{2}\left(z - \frac{q_0^2}{z}\right), \qquad \lambda(z) = \frac{1}{2}\left(z + \frac{q_0^2}{z}\right). \tag{2.13}$$

This implies that we can discuss the scattering problem on a standard z-plane instead of the two-sheeted Riemann surface by the inverse mapping. We define D^+, D^- and Σ on z-plane as

$$\Sigma = \mathbb{R} \cup i\mathbb{R}\backslash\{0\}, \qquad D^+ = \{z : \mathrm{Re}z\mathrm{Im}z > 0\}, \qquad D^- = \{z : \mathrm{Re}z\mathrm{Im}z < 0\}.$$

The two domains are shown in Fig. 1.

From these discussions, we can derive that

$$\begin{aligned}\mathrm{Im}(k(z)\lambda(z)) &= \mathrm{Im}\frac{z^4 - q_0^4}{4z^2} = \mathrm{Im}\frac{(|z|^4 + q_0^4)z^2 - 2q_0^4((\mathrm{Re}z)^2 - (\mathrm{Im}z)^2)}{4|z|^4} \\ &= \frac{1}{4|z|^4}(|z|^4 + q_0^4)\mathrm{Im}z^2 = \frac{1}{2|z|^4}(|z|^4 + q_0^4)\mathrm{Re}z\mathrm{Im}z,\end{aligned}$$

which implies that

$$\mathrm{Im}(k(z)\lambda(z))\begin{cases} = 0, & \text{as } z \in \Sigma \\ > 0, & \text{as } z \in D^+ \\ < 0, & \text{as } z \in D^- \end{cases}. \tag{2.14}$$

3. Jost solution

For eigenvalue $\pm i\lambda$, we can write the asymptotic eigenvector matrix as

$$Y_\pm = \begin{pmatrix} 1 & -\frac{iq_\pm}{z} \\ -\frac{iq_\pm^*}{z} & 1 \end{pmatrix} = I - \frac{i}{z}\sigma_3 Q_\pm, \tag{3.1}$$

so that $X_\pm$ and $T_\pm$ can be diagonalized by $Y_\pm$

$$X_\pm = Y_\pm(-ik\lambda\sigma_3)Y_\pm^{-1}, \qquad T_\pm = Y_\pm(-(2k^2 - q_0^2)ik\lambda\sigma_3)Y_\pm^{-1}. \tag{3.2}$$

Direct computation shows that

$$\det(Y_\pm) = 1 + \frac{q_0^2}{z^2} \triangleq \gamma, \tag{3.3}$$

and

$$Y_{\pm}^{-1}=\frac{1}{\gamma}\begin{pmatrix}1 & \frac{iq_{\pm}}{z}\\ \frac{iq_{\pm}^{*}}{z} & 1\end{pmatrix}=\frac{1}{\gamma}(I+\frac{i}{z}\sigma_3 Q_{\pm}),\qquad z\neq\pm iq_0. \tag{3.4}$$

Substituting (3.2) in (2.8), we immediately obtain

$$(Y_{\pm}^{-1}\psi)_x=-ik\lambda\sigma_3(Y_{\pm}^{-1}\psi),\qquad (Y_{\pm}^{-1}\psi)_t=-(2k^2-q_0^2)ik\lambda\sigma_3(Y_{\pm}^{-1}\psi),\qquad z\neq\pm iq_0, \tag{3.5}$$

from which we can derive the solution of the asymptotic spectral problem (2.8)

$$\psi(x,t,z)=\begin{cases}Y_{\pm}\mathrm{e}^{i\theta(z)\sigma_3}, & z\neq\pm iq_0,\\ I+(x-3q_0^2t)Y_{\pm}(z), & z=\pm iq_0,\end{cases} \tag{3.6}$$

where

$$\theta(x,t,z)=-k(z)\lambda(z)[x+(2k^2(z)-q_0^2)t].$$

For convenience, we will omit x and t dependence in $\theta(x,t,z)$ henceforth.

We define the Jost eigenfunctions $\phi_{\pm}(x,t,z)$ as the simultaneous solutions of both parts of the Lax pair so that

$$\phi_{\pm}=Y_{\pm}\mathrm{e}^{i\theta(z)\sigma_3}+o(1),\qquad x\to\pm\infty. \tag{3.7}$$

We introduce modified eigenfunctions by factorizing the asymptotic exponential oscillations

$$\mu_{\pm}=\phi_{\pm}\mathrm{e}^{-i\theta(z)\sigma_3}; \tag{3.8}$$

then, we have

$$\mu_{\pm}\sim Y_{\pm},\qquad x\to\pm\infty.$$

Meanwhile, $\mu_{\pm}$ acquire the equivalent Lax pair

$$(Y_{\pm}^{-1}\mu_{\pm})_x-ik\lambda[Y_{\pm}^{-1}\mu_{\pm},\sigma_3]=Y_{\pm}^{-1}\Delta X_{\pm}\mu_{\pm}, \tag{3.9}$$

$$(Y_{\pm}^{-1}\mu_{\pm})_t-ik\lambda(2k^2-q_0^2)[Y_{\pm}^{-1}\mu_{\pm},\sigma_3]=Y_{\pm}^{-1}\Delta T_{\pm}\mu_{\pm}, \tag{3.10}$$

where $\Delta X_{\pm}=X-X_{\pm}$ and $\Delta T_{\pm}=T-T_{\pm}$. These two equations can be written in full derivative form

$$d(\mathrm{e}^{-i\theta(z)\hat{\sigma}_3}Y_{\pm}^{-1}\mu_{\pm})=\mathrm{e}^{-i\theta(z)\hat{\sigma}_3}[Y_{\pm}^{-1}(\Delta X_{\pm}dx+\Delta T_{\pm}dt)\mu_{\pm}], \tag{3.11}$$

which leads to the Volterra integral equations

$$\mu_{\pm}(x,t,z)=\begin{cases}Y_{\pm}+\int_{\pm\infty}^{x}Y_{\pm}\mathrm{e}^{-ik\lambda(x-y)\hat{\sigma}_3}[Y_{\pm}^{-1}\Delta X_{\pm}(y,t)\mu_{\pm}(y,t,z)]\mathrm{d}y, & z\neq\pm iq_0,\\ Y_{\pm}+\int_{\pm\infty}^{x}[I+(x-y)X_{\pm}(z)]\Delta X_{\pm}(y,t)\mu_{\pm}(y,t,z)\mathrm{d}y, & z=\pm iq_0,\end{cases} \tag{3.12}$$

where we define $\mathrm{e}^{\alpha\hat{\sigma}}A:=\mathrm{e}^{\alpha\sigma}A\mathrm{e}^{-\alpha\sigma}$, for a matrix A.

Proposition 1. *Suppose $q(x,t)-q_{\pm}\in L^1(\mathbb{R}^{\pm})$, then the Volterra integral equation (3.12) has unique solutions $\mu_{\pm}(x,t,z)$ defined by (3.8) in $\Sigma_0:=\Sigma\setminus\{\pm iq_0\}$. Moreover, the columns $\mu_{-,1}$ and $\mu_{+,2}$ can be analytically extended to D^+ and continuously extended to $D^+\cup\Sigma_0$, while the columns $\mu_{+,1}$ and $\mu_{-,2}$ can be analytically extended to D^- and continuously extended to $D^-\cup\Sigma_0$, where $\mu_{\pm,j}(x,t,z)(j=1,2)$ denote the j-th column of $\mu_{\pm}$.*

Proof. We can define $\mu_{\pm}=(\mu_{\pm,1},\mu_{\pm,2})$ to rewrite columns of $\mu_{\pm}$. Since $x-y>0$, for μ_-, letting $W(x,z)=Y_-^{-1}\mu_-$, then the first column w of W is

$$w(x,z)=\begin{pmatrix}1\\0\end{pmatrix}+\int_{-\infty}^{x}G(x-y,z)\Delta X_-(y)Y_-(y)w(y,z)\mathrm{d}y, \tag{3.13}$$

where

$$G(x-y,z)=\mathrm{diag}(1,\mathrm{e}^{2ik\lambda(x-y)})Y_-^{-1}(z)=\frac{1}{\gamma}\begin{pmatrix}1 & \frac{iq_-}{z}\\ \frac{iq_-^*}{z}\mathrm{e}^{2ik\lambda(x-y)} & \mathrm{e}^{2ik\lambda(x-y)}\end{pmatrix}. \tag{3.14}$$

Now we introduce a Neumann series representation for w:

$$w(x,z)=\sum_{n=0}^{\infty}w^{(n)}, \tag{3.15}$$

with $w^{(0)}=\begin{pmatrix}1\\0\end{pmatrix}$, $w^{(n+1)}(x,z)=\int_{-\infty}^{x}C(x,y,z)w^{(n)}(y,z)\mathrm{d}y$, and where $C(x,y,z)=G(x-y,z)\Delta X_-Y_-$. Introducing the L^1 vector norm $\|w\|=|w_1|+|w_2|$ and the corresponding subordinate matrix norm $\|C\|$, we have

$$\|w^{(n+1)}(x,z)\|\le\int_{-\infty}^{x}\|C(x,y,z)\|\|w^{(n)}(y,z)\|\mathrm{d}y. \tag{3.16}$$

Note that $\|Y_\pm\|=1+q_0/|z|$ and $\|Y_\pm^{-1}\|=(1+q_0/|z|)/|1+q_0^2/z^2|$. The properties of the matrix norm imply

$$\begin{aligned}\|C(x,y,z)\|&\le\|\mathrm{diag}(1,\mathrm{e}^{2ik\lambda(x-y)})\|\|Y_-\|\|\Delta X_-(y)\|\|Y_-^{-1}\|\\&\le c(z)(1+\mathrm{e}^{-2\mathrm{Im}(k\lambda)(x-y)})(|k||q(y)-q_-|+\big||q|^2-q_0^2\big|),\end{aligned} \tag{3.17}$$

where $c(z)=\|Y_-\|\|Y_-^{-1}\|=(1+q_0/|z|)^2/|1+q_0^2/z^2|$. Recall that $\mathrm{Im}(k\lambda)>0$ for $z\in D^+$. For any $\epsilon>0$, let $D_\epsilon^+:=D^+\setminus(B_\epsilon(iq_0)\cup B_\epsilon(-iq_0))$, where $B_\epsilon(\pm iq_0)=\{z\in\mathbb{C}:|z\mp iq_0|<\epsilon q_0\}$. Then, we have

$$c_\epsilon=\max_{z\in D_\epsilon^+}c(z)=2+\frac{2}{\epsilon}. \tag{3.18}$$

Then, we prove that, for all $z\in D_\epsilon^+$ and for all $n\in\mathbb{N}$,

$$\|w^{(n)}(x,z)\|\le\frac{M^n(x)}{n!}, \tag{3.19}$$

where

$$M(x)=2c_\epsilon\int_{-\infty}^{x}(|k||q(y)-q_-|+\big||q|^2-q_0^2\big|)\mathrm{d}y. \tag{3.20}$$

We will prove the result by induction. The claim is trivially true for $n=0$. For all $z\in\overline{D^+}$ and for all $y\le x$, we have $1+\mathrm{e}^{-2\mathrm{Im}(k\lambda)(x-y)}\le 2$. If (3.19) holds for $n=j$, (3.16) implies

$$\|w^{(j+1)}(x,z)\|\le\frac{2c_\epsilon}{j!}\int_{-\infty}^{x}(\big||k|q(y)-q_-\big|+\big||q|^2-q_0^2)M^j(y)\mathrm{d}y=\frac{M^{j+1}(x)}{(j+1)!}. \tag{3.21}$$

Thus, for $\epsilon>0$, if $q(x)-q_-\in L^1(-\infty,a)$ for some $a\in\mathbb{R}$, then $|q(x)|^2-q_0^2\in L^1(-\infty,a)$ because of the boundedness of $q(x)$. The Neumann series converges absolutely and uniformly with respect to $x\in(-\infty,a)$ and $z\in D_\epsilon^+$. Since a uniformly convergent series of analytic functions converges to an analytic function, this demonstrates that the corresponding column of the Jost solution is analytic in this domain. □

Corollary 1. *Suppose $q(x,t)-q_\pm\in L^1(\mathbb{R}^\pm)$, then the Volterra integral equation (2.7) has unique solutions $\mu_\pm(x,t,z)$ defined by (3.7) in Σ_0. Moreover, the columns $\phi_{-,1}$ and $\phi_{+,2}$ can be analytically extended to D^+ and continuously extended to $D^+\cup\Sigma_0$, while the columns $\phi_{+,1}$ and $\phi_{-,2}$ can be analytically extended to D^- and continuously extended to $D^-\cup\Sigma_0$, where $\phi_{\pm,j}(x,t,z)(j=1,2)$ denote the j-th column of $\phi_\pm$.*

Proposition 2. *Suppose* $(1+|x|)(q(x,t)-q_\pm)\in L^1(\mathbb{R}^\pm)$, *then the Volterra integral equation* (3.12) *has unique solutions* $\mu_\pm(x,t,z)$ *defined by* (3.8) *in* Σ. *Besides, the columns* $\mu_{-,1}$ *and* $\mu_{+,2}$ *can be analytically extended to* D^+ *and continuously extended to* $D^+\cup\Sigma$, *while the columns* $\mu_{+,1}$ *and* $\mu_{-,2}$ *can be analytically extended to* D^- *and continuously extended to* $D^-\cup\Sigma$.

Proof. Note that at the branch points $z=\pm iq_0$, we have $\det(Y_\pm(x,t))=0$, which means that $Y_\pm(x,t)$ have no inverse at the branch points. However, if $(1+|x|)(q(x,t)-q_\pm)\in L^1(\mathbb{R})$, the integral equations have well-defined limits as $z\to\pm iq_0$. To see this, for $z\neq\pm iq_0$,

$$Y_\pm \mathrm{e}^{-ik\lambda(x-y)\sigma_3}Y_\pm^{-1}=\frac{1}{k\lambda}\sin(k\lambda(x-y))X_\pm+\cos(k\lambda(x-y))I. \tag{3.22}$$

As $z\to\pm iq_0$, $k\lambda\to 0$, the limit of the right-hand side is $I+(x-y)X_\pm(z)$, implying

$$\mu_\pm(x,t,z)=Y_\pm+\int\limits_{\pm\infty}^{x}[I+(x-y)X_\pm(z)]\Delta X_\pm(y,t)\mu_\pm(y,t,z)\mathrm{d}y, \qquad z=\pm iq_0. \tag{3.23}$$

Then, we have

$$\mu_{+,1}(x,t,z)=\begin{pmatrix}1\\ \mathrm{e}^{i\theta_+}\end{pmatrix}-\int\limits_{x}^{+\infty}[I+(x-y)X_+(z)]\Delta X_+(y,t)\mu_{+,1}(y,t,z)\mathrm{d}y, \qquad z=\pm iq_0. \tag{3.24}$$

where $\theta_+=\arg(q_+)$. Using the same technique in the proof of Proposition 1, we can finish the proof. $\square$

Corollary 2. *Suppose* $(1+|x|)(q(x,t)-q_\pm)\in L^1(\mathbb{R}^\pm)$, *then the Volterra integral equation* (2.7) *has unique solutions* $\phi_\pm(x,t,z)$ *defined by* (3.7) *in* Σ. *Besides, the columns* $\phi_{-,1}$ *and* $\phi_{+,2}$ *can be analytically extended to* D^+ *and continuously extended to* $D^+\cup\Sigma$, *while the columns* $\phi_{+,1}$ *and* $\phi_{-,2}$ *can be analytically extended to* D^- *and continuously extended to* $D^-\cup\Sigma$.

Lemma 1. *Consider an n-dimensional first-order homogeneous linear ordinary differential equation,* $\mathrm{d}y(x)/\mathrm{d}x=A(x)y(x)$, *on an interval* $\mathbb{D}\in\mathbb{R}$, *where* $A(x)$ *denotes a complex square matrix of order* n. *Let* Φ *be a matrix-valued solution of this equation. If the trace* $trA(x)$ *is a continuous function, then one has*

$$\det\Phi(x)=\det\Phi(x_0)\exp\left[\int\limits_{x_0}^{x} trA(\xi)\mathrm{d}\xi\right], \qquad x,x_0\in\mathbb{D}. \tag{3.25}$$

Proposition 3. *The Jost solutions* $\Phi(x,t,z)$ *are the simultaneous solutions of both parts of the Lax pair* (2.7).

4. Scattering matrix

Since $\mathrm{tr}X=\mathrm{tr}T=0$ in (2.7), then by using Abel formula, we have

$$(\det\phi_\pm)_x=(\det\phi_\pm)_t=0, \qquad \det(\mu_\pm)=\det(\phi_\pm \mathrm{e}^{-i\theta(z)\sigma_3})=\det(\phi_\pm),$$

so that $(\det\mu_\pm)_x=(\det\mu_\pm)_t=0$, which means $\det(\mu_\pm)$ is independent with x,t. Furthermore, we know that $\mu_\pm$ is invertible from

$$\det\mu_\pm=\lim_{x\to\pm\infty}\det(\mu_\pm)=\det Y_\pm=\gamma\neq 0, \qquad x,t\in\mathbb{R}, \qquad z\in\Sigma_0. \tag{4.1}$$

Since $\phi_\pm$ are two fundamental matrix solutions of the linear Lax pair (2.7), there exists a relation between ϕ_+ and ϕ_-

$$\phi_+(x,t,z)=\phi_-(x,t,z)S(z), \qquad x,t\in\mathbb{R}, \qquad z\in\Sigma_0, \tag{4.2}$$

where $S(z)$ is called scattering matrix and (4.1) implies that $\det S(z) = 1$. Letting $S(z) = (s_{ij})$, for the individual columns

$$\phi_{+,1} = s_{11}\phi_{-,1} + s_{21}\phi_{-,2}, \qquad \phi_{+,2} = s_{12}\phi_{-,1} + s_{22}\phi_{-,2}. \tag{4.3}$$

By using (4.2), we obtain

$$s_{11}(z) = \frac{\mathrm{Wr}(\phi_{+,1}, \phi_{-,2})}{\gamma}, \qquad s_{12}(z) = \frac{\mathrm{Wr}(\phi_{+,2}, \phi_{-,2})}{\gamma}, \tag{4.4}$$

$$s_{21}(z) = \frac{\mathrm{Wr}(\phi_{-,1}, \phi_{+,1})}{\gamma}, \qquad s_{22}(z) = \frac{\mathrm{Wr}(\phi_{-,1}, \phi_{+,2})}{\gamma}. \tag{4.5}$$

Proposition 4. *Suppose $q(x,t) - q_\pm \in L^1(\mathbb{R}^\pm)$. Then, s_{11} can be analytically extended to D^- and continuously extended to $D^- \cup \Sigma_0$, while s_{22} can be analytically extended to D^+ and continuously extended to $D^+ \cup \Sigma_0$. Moreover, s_{12} and s_{21} are continuous in Σ_0.*

Corollary 3. *Suppose $(1 + |x|)(q(x,t) - q_\pm) \in L^1(\mathbb{R}^\pm)$. Then, $\lambda(z)s_{11}(z)$ can be analytically extended to D^- and continuously extended to $D^- \cup \Sigma$, while s_{22} can be analytically extended to D^+ and continuously extended to $D^+ \cup \Sigma$. Moreover, $\lambda(z)s_{12}$ and $\lambda(z)s_{21}$ are continuous in Σ.*

Note that we cannot exclude the possible existence of zeros for $s_{11}(z)$ and $s_{22}(z)$ along Σ_0. To solve the Riemann–Hilbert problem, we restrict our consideration to potentials without spectral singularities, i.e., $s_{11}(z) \neq 0$, $s_{22}(z) \neq 0$ for $z \in \Sigma$. Besides, we assume that the scattering coefficients are continuous at the branch points. The reflection coefficients which will be needed in the inverse problem are

$$\rho(z) = \frac{s_{21}}{s_{11}}, \qquad \tilde{\rho}(z) = \frac{s_{12}}{s_{22}}. \tag{4.6}$$

5. Symmetry

For the GI equation with nonzero boundary, we not only need to deal with the map $k \mapsto k^*$, but also need to pay attention to the sheets of the Riemann surface. We can see from the Riemann surface that the transformation $z \mapsto z^*$ implies $(k, \lambda) \mapsto (k^*, \lambda^*)$ and $z \mapsto -q_0^2/z$ implies $(x, \lambda) \mapsto (k, -\lambda)$. Therefore, we would like to discuss the symmetries in the following way.

Proposition 5. *The Jost solution, scattering matrix and reflection coefficients satisfy the following reduction conditions on z-plane*

- *The first symmetry reduction*

$$\phi_\pm(x,t,z) = \sigma_2\phi_\pm^*(x,t,z^*)\sigma_2, \qquad S(z) = \sigma_2 S^*(z^*)\sigma_2, \qquad \rho(z) = -\tilde{\rho}^*(z^*), \tag{5.1}$$

where $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

- *The second symmetry reduction*

$$\phi_\pm(x,t,z) = \sigma_1\phi_\pm^*(x,t,-z^*)\sigma_1, \qquad S(z) = \sigma_1 S^*(-z^*)\sigma_1, \qquad \rho(z) = \tilde{\rho}^*(-z^*), \tag{5.2}$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

- *The third symmetry reduction*

$$\phi_\pm(x,t,z) = -\frac{i}{z}\phi_\pm(x,t,-\frac{q_0^2}{z})\sigma_3 Q_\pm, \tag{5.3}$$

$$S(z) = (\sigma_3 Q_-)^{-1} S(-\frac{q_0^2}{z})\sigma_3 Q_+, \qquad \rho(z) = \frac{q_-}{q_-^*}\tilde{\rho}(-\frac{q_0^2}{z}). \tag{5.4}$$

Proof. Let $\phi(x,t,z)$ be a solution of the scattering problem (2.7), so we have

$$\phi_x = X\phi = (-ik^2\sigma_3 + \frac{i}{2}(|q|^2 - q_0^2)\sigma_3 + kQ)\phi. \tag{5.5}$$

Replacing z with z^*, conjugating both sides at the same time and then premultiplying both sides by σ_2, we get the equation

$$\sigma_2\phi_x^*(x,t,z^*) = \left(ik^{*2}(z^*)\sigma_2\sigma_3 - \frac{i}{2}(|q|^2 - q_0^2)\sigma_2\sigma_3 + k^*(z^*)\sigma_2 Q^*\right)\phi^*(x,t,x^*). \tag{5.6}$$

Because of $k^*(z^*) = k(z)$, $\sigma_2\sigma_3\sigma_2^{-1} = -\sigma_3$, and $\sigma_2 Q^*\sigma_2^{-1} = Q$, the above equation then becomes

$$\sigma_2\phi_x^*(x,t,z^*) = \left(-ik^2(z)\sigma_3 + \frac{i}{2}(|q|^2 - q_0^2)\sigma_3 + k(z)Q\right)(\sigma_2\phi^*(z^*)). \tag{5.7}$$

Hence, $\sigma_2\phi^*(x,t,z^*)$ is also the solution of the scattering problem (2.7), so is $\sigma_2\phi^*(x,t,z^*)C$, where C is an arbitrary 2×2 matrix.

Letting $z \in \mathbf{C}\setminus\Sigma$, with $\theta^*(z^*) = \theta(z)$ and $\sigma_2 Y_\pm^*(z^*) = \sigma_2(I + i\sigma_3 Q_\pm^*/z) = Y_\pm\sigma_2$, we can derive that

$$\sigma_2\phi_\pm^*(x,t,z^*)\sigma_2 = -Y_\pm(x,t,x)\mathrm{e}^{i\theta(z)\sigma_3} + O(1), \qquad x\to\pm\infty. \tag{5.8}$$

From the uniqueness of the solution of the scattering problem, we can get the conclusion that

$$-\sigma_2\phi_\pm^*(x,t,z^*)\sigma_2 = \phi_\pm. \tag{5.9}$$

This is exactly what the first equation of (5.1) says. Using similar method, with symmetries $\theta^*(-z^*) = \theta(z)$, $\sigma_1\sigma_3\sigma_1^{-1} = -\sigma_3$ and $\sigma_1 Q^*\sigma_1 = -Q$, we can easily get the second symmetry of $\phi_\pm(x,t,z)$.

Next, we show the equation in (5.2). If $\phi(x,t,z)$ is a solution of the scattering problem (2.7), since

$$k\left(-\frac{q_0^2}{z}\right) = \frac{1}{2}\left(-\frac{q_0^2}{z} + z\right) = k(z), \tag{5.10}$$

so is $\phi(x,t,-\frac{q_0^2}{z})$ and $\phi(x,t,-\frac{q_0^2}{z})C$, for any 2×2 matrix C independent of t. With $\theta(-\frac{q_0^2}{z}) = -\theta(z)$, it is apparent that

$$\phi_\pm\left(x,t,-\frac{q_0^2}{z}\right)C = Y_\pm\left(-\frac{q_0^2}{z}\right)\mathrm{e}^{-i\theta(z)\sigma_3}C. \tag{5.11}$$

Note that

$$-\frac{i}{z}Y_\pm\left(-\frac{q_0^2}{z}\right)\mathrm{e}^{-i\theta\sigma_3}\sigma_3 Q_\pm = Y_\pm(z)\mathrm{e}^{i\theta\sigma_3},$$

and taking $C = -\frac{i}{z}\sigma_3 Q_\pm$, we get the symmetry

$$-\frac{i}{z}\phi\left(x,t,-\frac{q_0^2}{z}\right)\sigma_3 Q_\pm = Y_\pm\mathrm{e}^{i\theta\sigma_3}.$$

What will come next are the symmetries of scattering matrix. For the individual columns, the above symmetries come to

$$\phi_{\pm,1}(x,t,z) = i\sigma_2\phi_{\pm,2}^*(x,t,z^*), \qquad \phi_{\pm,2}(x,t,z) = -i\sigma_2\phi_{\pm,1}^*(x,t,z^*), \tag{5.12}$$

$$\phi_{\pm,1}(x,t,z) = -\frac{i}{z}q_\pm^*\phi_{\pm,2}\left(-\frac{q_0^2}{z}\right), \qquad \phi_{\pm,2}(x,t,z) = -\frac{i}{z}q_\pm\phi_{\pm,1}\left(-\frac{q_0^2}{z}\right). \tag{5.13}$$

Substituting the first one of (5.1) into (4.2), we obtain a symmetry of scattering matrix

$$S^*(z^*) = \sigma_2 S(z)\sigma_2, \tag{5.14}$$

closely followed by the relations between the scattering coefficients

$$s_{11}(z) = s_{22}^*(z^*), \qquad s_{12}(z) = -s_{21}^*(z^*). \tag{5.15}$$

Similarly, we can get $S(z) = \sigma_1 S^*(-z^*)\sigma_1$, which leads to

$$s_{11}(z) = s_{22}^*(-z^*), \qquad s_{12}(z) = s_{21}^*(-z^*). \tag{5.16}$$

Substituting (5.2) in (4.2), we obtain another symmetry of the scattering matrix

$$S\left(-\frac{q_0^2}{z}\right) = \sigma_3 Q_- S(z)(\sigma_3 Q_+)^{-1}, \tag{5.17}$$

which is also followed by the relations between the scattering coefficients

$$s_{11}(z) = \frac{q_+^*}{q_-^*} s_{22}\left(-\frac{q_0^2}{z}\right), \qquad s_{12}(z) = \frac{q_+}{q_-^*} s_{21}\left(-\frac{q_0^2}{z}\right), \tag{5.18}$$

$$s_{21}(z) = \frac{q_+^*}{q_-} s_{12}\left(-\frac{q_0^2}{z}\right), \qquad s_{22}(z) = \frac{q_+}{q_-} s_{11}\left(-\frac{q_0^2}{z}\right). \tag{5.19}$$

Combining (5.14) with (5.17), we then get

$$S^*(z^*) = \sigma_2(\sigma_3 Q_-)^{-1} S\left(-\frac{q_0^2}{z}\right)\sigma_3 Q_+ \sigma_2. \tag{5.20}$$

Elementwise,

$$s_{11}^*(z^*) = \frac{q_+}{q_-} s_{11}\left(-\frac{q_0^2}{z}\right), \qquad s_{12}^*(z^*) = -\frac{q_+^*}{q_-} s_{12}\left(-\frac{q_0^2}{z}\right), \tag{5.21}$$

$$s_{21}^*(z^*) = -\frac{q_+}{q_-^*} s_{21}\left(-\frac{q_0^2}{z}\right), \qquad s_{22}^*(z^*) = \frac{q_+^*}{q_-^*} s_{22}\left(-\frac{q_0^2}{z}\right). \tag{5.22}$$

Finally, all the above symmetries then give the symmetries for the reflection coefficients

$$\rho(z) = \tilde{\rho}^*(-z^*) = -\tilde{\rho}^*(z^*) = \frac{q_-}{q_-^*}\tilde{\rho}(-\frac{q_0^2}{z}) = -\frac{q_-^*}{q_-}\rho^*(-\frac{q_0^2}{z^*}). \tag{5.23}$$

From (5.23), we can also get an important symmetry

$$\rho(z) = -\rho(-z). \tag{5.24}$$

So we have done the proof. □

6. Discrete spectrum and residue conditions

The discrete spectrum of the scattering problem is the set of all values $z \in \mathbb{C} \setminus \Sigma$, for which eigenfunctions exist in $L^2(\mathbb{R})$. We would like to show that these values are the zeros of $s_{11}(z)$ in D^- and those of $s_{22}(z)$ in D^+.

We can show that the uniformization transformation (2.13) changes the segment $[-iq_0, iq_0]$ on k-plane into the circle $|z| = q_0$ on z-plane. We suppose that s_{22} has N_1 simple zeros $z_1, \ldots, z_{N_1}$ in $D^+ \cap \{z \in \mathbb{C} : \mathrm{Im} z > 0, |z| > q_0\}$, and N_2 simple zeros $w_1, \ldots, w_m$ in $\{z = q_0 \mathrm{e}^{i\varphi} : 0 < \varphi < \frac{\pi}{2}\}$, that is, $s_{22}(z) = 0$ and ${s_{22}}'(z) \neq 0$ if z is a simple zero of s_{22}. Then, symmetries (5.1)–(5.19) imply that

$$s_{22}(\pm z_n) = 0 \Leftrightarrow s_{11}^*(\pm z_n^*) = 0 \Leftrightarrow s_{11}\left(\pm\frac{q_0^2}{z_n}\right) = 0 \Leftrightarrow s_{22}\left(\pm\frac{q_0^2}{z_n^*}\right) = 0, \qquad n = 1, \ldots, N_1, \tag{6.1}$$

and

$$s_{22}(\pm w_m) = 0 \Leftrightarrow s_{11}^*(\pm w_m^*) = 0. \qquad m = 1, \ldots, N_2. \tag{6.2}$$

Therefore, the discrete spectrum is the set

$$Z = \left\{\pm z_n, \pm z_n^*, \pm\frac{q_0^2}{z_n}, \pm\frac{q_0^2}{z_n^*}\right\}_{n=1}^{N_1} \bigcup \left\{\pm w_m, \pm w_m^*\right\}_{m=1}^{N_2}, \tag{6.3}$$

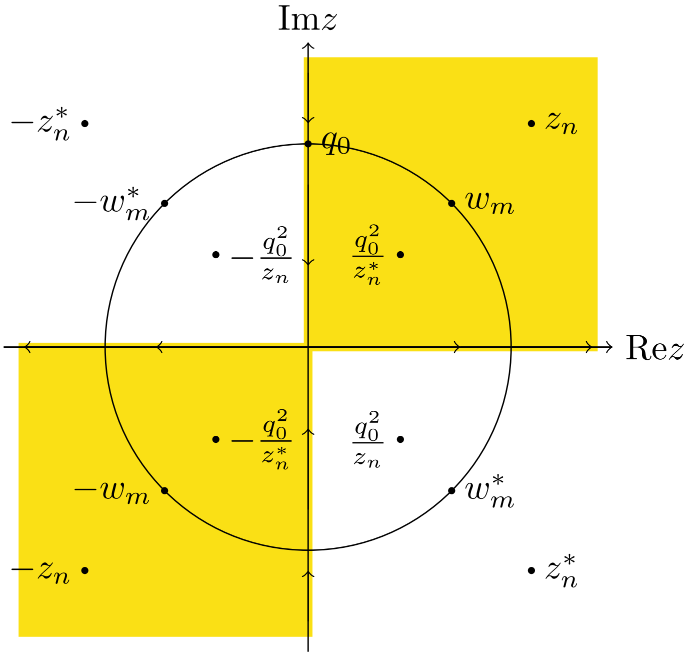

Fig. 2. Distribution of the discrete spectrum and the contours for the Riemann–Hilbert problem on complex z-plane

which can be seen in Fig. 2.

If $s_{11}(z) = 0$ at $z = z_n^*$, from the first of (4.4), the eigenfunction $\phi_{+,1}$ and $\phi_{-,2}$ must be proportional at z_n^*

$$\phi_{+,1}(x,t,z_n^*) = b_n\phi_{-,2}(x,t,z_n^*), \qquad b_n \neq 0 \text{ independent of } x,t,z. \tag{6.4}$$

Then, we would like to derive the residue conditions which will be needed for the Riemann–Hilbert problem.

Owing to (6.4), we have $\mu_{+,1}(x,t,z_n^*) = b_n e^{-2i\theta(z_n^*)}\mu_{-,2}(x,t,z_n^*)$. As a result,

$$\operatorname*{Res}_{z=z_n^*}\left[\frac{\mu_{+,1}(x,t,z)}{s_{11}(z)}\right] = \lim_{z\to z_n^*}(z-z_n^*)\frac{\mu_{+,1}(x,t,z)}{s_{11}(z)} = C_n e^{-2i\theta(z_n^*)}\mu_{-,2}(x,t,z_n^*), \tag{6.5}$$

where $C_n = \frac{b_n}{s_{11}{}'(z_n^*)}$.

If $s_{11}(-z_n^*) = 0$, with the symmetries (5.2), (5.1), and (4.4), we get

$$\phi_{+,1}(x,t,-z_n^*) = -b_n\phi_{-,2}(x,t,-z_n^*) \tag{6.6}$$

$$\phi_{-,2}(x,t,-z_n^*) = -\sigma_3\phi_{-,2}(x,t,z_n^*), \tag{6.7}$$

which also mean

$$\mu_{+,1}(x,t,-z_n^*) = -b_n e^{-2i\theta(-z_n^*)}\mu_{-,2}(x,t,-z_n^*) \tag{6.8}$$

$$\mu_{-,2}(x,t,-z_n^*) = -\sigma_3\mu_{-,2}(x,t,z_n^*). \tag{6.9}$$

Owing to (6.1), we have $(s_{11}(-z_n^*))' = -s_{11}'(z_n^*)$. It is obvious that

$$\operatorname*{Res}_{z=-z_n^*}\left[\frac{\mu_{+,1}(x,t,z)}{s_{11}(z)}\right] = \frac{-b_n e^{-2i\theta(-z_n^*)}\mu_{-,2}(x,t,-z_n^*)}{-s_{11}'(z_n^*)} = -C_n e^{-2i\theta(z_n^*)}\sigma_3\mu_{-,2}(x,t,z_n^*), \tag{6.10}$$

because of $\theta(-z_n^*) = \theta(z_n^*)$.

If $s_{22}(z_n)=0$, then

$$\phi_{+,2}(x,t,z_n)=\tilde{b}_n\phi_{-,1}(x,t,z_n); \tag{6.11}$$

in other words, it is $\mu_{+,2}(x,t,z_n)=\tilde{b}_n\mathrm{e}^{2i\theta(z_n)}\mu_{-,1}(x,t,z_n)$. In that way, we can derive that

$$\operatorname*{Res}_{z=z_n}\left[\frac{\mu_{+,2}(x,t,z)}{s_{22}(z)}\right]=\tilde{C}_n\mathrm{e}^{2i\theta(z_n)}\mu_{-,1}(x,t,z_n), \tag{6.12}$$

where $\tilde{C}_n=\frac{\tilde{b}_n}{s'_{22}(z_n)}$. If $s_{22}(-z_n)=0$, also from (5.1), (5.2) and (4.5) we can get

$$\phi_{-,1}(x,t,-z_n)=\sigma_3\phi_{-,1}(x,t,z_n), \tag{6.13}$$

$$\phi_{+,2}(x,t,-z_n)=-\tilde{b}_n\phi_{-,1}(x,t,-z_n), \tag{6.14}$$

which also mean

$$\mu_{-,1}(x,t,-z_n)=\sigma_3\mu_{-,1}(x,t,z_n), \tag{6.15}$$

$$\mu_{+,2}(x,t,-z_n)=-\tilde{b}_n\mathrm{e}^{2i\theta(-z_n)}\mu_{-,1}(x,t,-z_n). \tag{6.16}$$

With (6.1), we have $(s_{22}(-z_n))'=-s'_{22}(z_n)$. Similarly, we can get the residue

$$\operatorname*{Res}_{z=-z_n}\left[\frac{\mu_{+,2}(x,t,z)}{s_{22}(z)}\right]=\tilde{C}_n\mathrm{e}^{2i\theta(z_n)}\sigma_3\mu_{-,1}(x,t,z_n). \tag{6.17}$$

Because of the symmetries, we can get the relations between the norming constants. Using the two equations in (5.12) and comparing with (6.11), we know that $\tilde{b}_n=-b_n^*$. It is clear that (6.1) implies $s'_{22}(z_n)=(s_{11}^*(z_n^*))'$, so we have

$$\tilde{C}_n=\frac{\tilde{b}_n}{s'_{22}(z_n)}=-\frac{b_n^*}{(s_{11}^*(z_n^*))'}=-C_n^*. \tag{6.18}$$

Substituting (5.13) in (6.4) and (6.11), we obtain

$$\phi_{+,2}\left(x,t,-\frac{q_0^2}{z_n^*}\right)=\frac{q_-}{q_+^*}b_n\phi_{-,1}\left(-\frac{q_0^2}{z_n^*}\right), \tag{6.19}$$

$$\phi_{+,1}\left(x,t,-\frac{q_0^2}{z_n}\right)=\frac{q_-^*}{q_+}\tilde{b}_n\phi_{-,2}\left(-\frac{q_0^2}{z_n}\right), \tag{6.20}$$

From (5.13), (6.6) and (6.14), we get

$$\phi_{+,2}\left(x,t,\frac{q_0^2}{z_n^*}\right)=-\frac{q_-}{q_+^*}b_n\phi_{-,1}\left(\frac{q_0^2}{z_n^*}\right), \tag{6.21}$$

$$\phi_{+,1}\left(x,t,\frac{q_0^2}{z_n}\right)=-\frac{q_-^*}{q_+}\tilde{b}_n\phi_{-,2}\left(\frac{q_0^2}{z_n}\right). \tag{6.22}$$

Furthermore, from the symmetries of the scattering coefficients (5.17) and (5.20), we get the relation

$$s_{11}\left(-\frac{q_0^2}{z_n}\right)=\frac{q_-}{q_+}s_{11}^*(z_n^*). \tag{6.23}$$

By differentiating the above equation, we have

$$s'_{11}\left(-\frac{q_0^2}{z_n}\right)=\left(\frac{z_n}{q_0}\right)^2\frac{q_-}{q_+}(s_{11}^*(z_n^*))'. \tag{6.24}$$

Similarly,

$$s'_{11}\left(\frac{q_0^2}{z_n}\right)=-\left(\frac{z_n}{q_0}\right)^2\frac{q_-}{q_+}(s_{11}^*(z_n^*))', \tag{6.25}$$

$$s'_{22}\left(-\frac{q_0^2}{z_n^*}\right)=\left(\frac{z_n^*}{q_0}\right)^2\left(\frac{q_-}{q_+}\right)^*(s_{22}^*(z_n))', \tag{6.26}$$

$$s'_{22}\left(\frac{q_0^2}{z_n^*}\right)=-\left(\frac{z_n^*}{q_0}\right)^2\left(\frac{q_-}{q_+}\right)^*(s_{22}^*(z_n))', \tag{6.27}$$

Finally, combining the above relations, we get

$$\operatorname*{Res}_{z=-\frac{q_0^2}{z_n}}\left[\frac{\mu_{+,1}(x,t,z)}{s_{11}(z)}\right]=C_{N_1+n}\mathrm{e}^{-2i\theta(-\frac{q_0^2}{z_n})}\mu_{-,2}\left(x,t,-\frac{q_0^2}{z_n}\right), \tag{6.28}$$

$$\operatorname*{Res}_{z=-\frac{q_0^2}{z_n^*}}\left[\frac{\mu_{+,2}(x,t,z)}{s_{22}(z)}\right]=\tilde{C}_{N_1+n}\mathrm{e}^{2i\theta\left(-\frac{q_0^2}{z_n^*}\right)}\mu_{-,1}\left(x,t,-\frac{q_0^2}{z_n^*}\right), \tag{6.29}$$

$$\operatorname*{Res}_{z=\frac{q_0^2}{z_n}}\left[\frac{\mu_{+,1}(x,t,z)}{s_{11}(z)}\right]=-C_{N_1+n}\mathrm{e}^{-2i\theta\left(-\frac{q_0^2}{z_n}\right)}\sigma_3\mu_{-,2}\left(x,t,-\frac{q_0^2}{z_n}\right), \tag{6.30}$$

$$\operatorname*{Res}_{z=\frac{q_0^2}{z_n^*}}\left[\frac{\mu_{+,2}(x,t,z)}{s_{22}(z)}\right]=\tilde{C}_{N_1+n}\mathrm{e}^{2i\theta\left(-\frac{q_0^2}{z_n^*}\right)}\sigma_3\mu_{-,1}\left(x,t,-\frac{q_0^2}{z_n^*}\right). \tag{6.31}$$

where

$$C_{N_1+n}=\frac{q_-^*}{q_-}\left(\frac{q_0}{z_n}\right)^2\tilde{C}_n,\qquad \tilde{C}_{N_1+n}=\left(\frac{q_0}{z_n^*}\right)^2\frac{q_-}{q_-^*}C_n, \tag{6.32}$$

with the relation

$$\tilde{C}_{N_1+n}=-C^*_{N_1+n}, \tag{6.33}$$

for $n=1,\ldots,N_1$.

Analogously, we get the residue conditions at $\pm w_m$ and $\pm w_m^*$,

$$\operatorname*{Res}_{z=w_m^*}\frac{\mu_{+,1}(x,t,z)}{s_{11}(z)}=C_{2N_1+m}\mathrm{e}^{-2i\theta(w_m^*)}\mu_{-,2}(x,t,w_m^*), \tag{6.34}$$

$$\operatorname*{Res}_{z=-w_m^*}\frac{\mu_{+,1}(x,t,z)}{s_{11}(z)}=-C_{2N_1+m}\mathrm{e}^{-2i\theta(w_m^*)}\sigma_3\mu_{-,2}(x,t,w_m^*), \tag{6.35}$$

$$\operatorname*{Res}_{z=w_m}\frac{\mu_{+,2}(x,t,z)}{s_{22}(z)}=\tilde{C}_{2N_1+m}\mathrm{e}^{2i\theta(w_m)}\mu_{-,1}(x,t,w_m), \tag{6.36}$$

$$\operatorname*{Res}_{z=-w_m}\frac{\mu_{+,2}(x,t,z)}{s_{22}(z)}=\tilde{C}_{2N_1+m}\mathrm{e}^{2i\theta(w_m)}\sigma_3\mu_{-,1}(x,t,w_m), \tag{6.37}$$

where $C_{2N_1+m}=\frac{b_{2N_1+m}}{s'_{11}(w_m^*)}$, $\tilde{C}_{2N_1+m}=-C^*_{2N_1+m}$ and b_{2N_1+m} is independent of x,t for $m=1,\ldots,N_2$.

7. Asymptotic behaviors

To solve the Riemann–Hilbert problem in the next section, it is necessary to discuss the asymptotic behaviors of the modified Jost solutions and scattering matrix as $z\to\infty$ and $z\to 0$ by the standard Wentzel–Kramers–Brillouin (WKB) expansions.

Proposition 6. *The asymptotic behaviors for the modified Jost solutions are given as*

$$\mu_\pm(x,t,z)=I+o(z^{-1}),\qquad z\to\infty, \tag{7.1}$$

$$\mu_\pm(x,t,z)=-\frac{i}{z}\sigma_3Q_\pm+o(1),\qquad z\to 0. \tag{7.2}$$

Proof. We consider the following ansatz for the expansions of the modified Jost solutions $\mu_\pm(x,t,z)$ as $z\to\infty$ and $z\to 0$

$$\mu_\pm(x,t,z)=\mu_\pm^{(0)}(x,t)+\frac{\mu_\pm^{(1)}(x,t)}{z}+\frac{\mu_\pm^{(2)}(x,t)}{z^2}+o(z^{-3}),\qquad \text{as } z\to\infty, \tag{7.3}$$

$$\mu_\pm(x,t,z)=\frac{\tilde{\mu}_\pm^{(-1)}(x,t)}{z}+\tilde{\mu}_\pm^{(0)}(x,t)+\tilde{\mu}_\pm^{(1)}(x,t)z+o(z^2),\qquad \text{as } z\to 0. \tag{7.4}$$

Substitute the above expansions into the Lax equation (3.9), and let A_d and A_o denote, respectively, the diagonal and off-diagonal parts of the matrix A. As $z\to\infty$, we have

$$\left[\left(I+\frac{i}{z}\sigma_3Q_\pm)(\mu_\pm^{(0)}+\frac{\mu_\pm^{(1)}}{z}+\frac{\mu_\pm^{(2)}}{z^2}+\cdots\right)\right]_x=\frac{i}{4}\left(z^2-\frac{q_0^4}{z^2}\right)\left[\left(I+\frac{i}{z}\sigma_3Q_\pm\right)\left(\mu_\pm^{(0)}+\frac{\mu_\pm^{(1)}}{z}+\frac{\mu_\pm^{(2)}}{z^2}+\cdots\right),\sigma_3\right] \tag{7.5}$$

$$+(I+\frac{i}{z}\sigma_3Q_\pm)\Big[\frac{i}{2}(|q|^2-q_0^2)\sigma_3+\frac{1}{2}(z-\frac{q_0^2}{z})\Delta Q_\pm\Big](\mu_\pm^{(0)}+\frac{\mu_\pm^{(1)}}{z}+\frac{\mu_\pm^{(2)}}{z^2}+\cdots). \tag{7.6}$$

By matching the $O(z^2)$ terms, we obtain that

$$[\mu_\pm^{(0)},\sigma_3]=0,$$

which means that $\mu_\pm^{(0)}$ is a diagonal matrix. We record it as

$$\mu_\pm^{(0)}=\begin{pmatrix} a(x) & 0\\ 0 & b(x)\end{pmatrix}.$$

By matching the $O(z)$ terms, we obtain that

$$\frac{i}{4}[\mu_\pm^{(1)},\sigma_3]-\frac{1}{4}[\sigma_3Q_\pm\mu_\pm^{(0)},\sigma_3]+\frac{1}{2}\Delta Q_\pm\mu_\pm^{(0)}=0; \tag{7.7}$$

then, we can get the off-diagonal part of $\mu_\pm^{(1)}$

$$\mu_{\pm,o}^{(1)}=\begin{pmatrix} 0 & -ib(x)q\\ -ia(x)q^* & 0\end{pmatrix}. \tag{7.8}$$

By matching the $O(1)$ terms, we obtain that

$$\mu_{\pm,x}^{(0)}=\frac{i}{4}[\mu_\pm^{(2)},\sigma_3]+\frac{i}{4}[i\sigma_3Q_\pm\mu_\pm^{(0)},\sigma_3]+\frac{i}{2}(|q|^2-q_0^2)\sigma_3\mu_\pm^{(0)}+\frac{1}{2}\Delta Q_\pm\mu_\pm^{(1)}+\frac{i}{2}\sigma_3Q_\pm\mu_\pm^{(0)}. \tag{7.9}$$

The left-hand side of (7.9) is a diagonal matrix, and the first two parts of the right-hand side are off-diagonal matrix, so we can just calculate the last three parts in the right-hand side. By calculating, we get

$$\begin{pmatrix} a_x & 0\\ 0 & b_x\end{pmatrix}=0, \tag{7.10}$$

so that

$$\mu_\pm^{(0)}=C, \tag{7.11}$$

where C is a constant matrix. To find out the exact value of C, we see that

$$\lim_{x\to\pm\infty}\mu_\pm=Y_\pm=I-\frac{i}{z}\sigma_3Q_\pm=\lim_{x\to\pm\infty}(\mu_\pm^{(0)}+\cdots), \tag{7.12}$$

so that $C=I$. Therefore, we get the asymptotic behavior of the modified Jost solution

$$\mu_\pm(x,t,z)=I+O(z^{-1}),\qquad z\to\infty,$$

When $z\to 0$, by matching the $O(z^{-4})$ terms, we obtain that

$$[\sigma_3Q_\pm\tilde{\mu}_\pm^{(-1)},\sigma_3]=0, \tag{7.13}$$

that is to say the diagonal part of $\mu_\pm^{(-1)}$ is 0, so we can record it as

$$\tilde{\mu}_\pm^{(-1)} = \begin{pmatrix} & \tilde{a}(x) \\ \tilde{b}(x) & \end{pmatrix}. \tag{7.14}$$

By matching the $O(z^{-3})$ terms, we obtain that

$$\tilde{\mu}_{\pm,d}^{(0)} = \begin{pmatrix} i\tilde{b}\frac{q}{q_0^2} & \\ & i\tilde{a}\frac{q^*}{q_0^2}. \end{pmatrix} \tag{7.15}$$

By matching the $O(z^{-2})$ terms, we obtain that $\tilde{\mu}_\pm^{(-1)} = \tilde{C}$, where $\tilde{C}$ is a constant matrix. From the expansion (7.4), we have

$$\lim_{x\to\pm\infty} z\tilde{\mu}_\pm = zY_\pm = z\left(I - \frac{i}{z}\sigma_3 Q_\pm\right) = \lim_{x\to\pm\infty}(\tilde{\mu}_\pm^{(-1)} + z\tilde{\mu}_\pm^{(0)} + \cdots). \tag{7.16}$$

Thereby, $\tilde{C} = -i\sigma_3 Q_\pm$ and $\tilde{\mu}_\pm^{(-1)} = -i\sigma_3 Q_\pm$. Finally, we get the asymptotic behavior

$$\mu_\pm(x,t,z) = -\frac{i}{z}\sigma_3 Q_\pm + O(1), \qquad z \to 0.$$

□

Inserting the above asymptotic behaviors for the modified Jost eigenfunctions into the Wronskian representation (4.4) and (4.5), with a little calculations, we get the asymptotic behaviors of the scattering matrix.

Proposition 7. *The asymptotic behaviors of the scattering matrix are*

$$S(z) = I + O(z^{-1}), \qquad z \to \infty, \tag{7.17}$$

$$S(z) = \mathrm{diag}\left(\frac{q_-}{q_+}, \frac{q_+}{q_-}\right) + O(z), \qquad z \to 0. \tag{7.18}$$

Proof. From the representation of the scattering coefficients (4.4), (4.5) and the asymptotic behaviors, we have

$$\begin{aligned} s_{11} = \frac{\mathrm{Wr}(\phi_{+,1}, \phi_{-,2})}{\gamma} &= \det\begin{pmatrix} 1 + O(z^{-1}) & O(z^{-1}) \\ O(z^{-1}) & 1 + O(z^{-1}) \end{pmatrix}\left(1 - \frac{q_0^2}{z^2} + \frac{q_0^4}{z^4} + \cdots\right) \\ &= (1 + O(z^{-1}))\left(1 - \frac{q_0^2}{z^2} + \frac{q_0^4}{z^4} + \cdots\right) \\ &= 1 + O(z^{-1}). \end{aligned}$$

Similarly,

$$s_{22} = \frac{\mathrm{Wr}(\phi_{-,1}, \phi_{+,2})}{\gamma} = 1 + O(z^{-1}).$$

Hence, we have the asymptotic behavior (7.17).

As $z \to 0$

$$\begin{aligned} s_{11} = \frac{\mathrm{Wr}(\phi_{+,1}, \phi_{-,2})}{\gamma} &= \det\begin{pmatrix} O(1) & -\frac{i}{z}q_- + O(1) \\ -\frac{i}{z}q_+^* + O(1) & O(1) \end{pmatrix}\left(\frac{z^2}{q_0^2} - \frac{z^4}{q_0^4} + \cdots\right) \\ &= \frac{q_-}{q_+} + O(z). \\ s_{22} &= \frac{q_+}{q_-} + O(z). \end{aligned}$$

Therefore, we obtain the asymptotic behavior (7.18). □

8. Riemann–Hilbert problem

As we all know, the equation (4.2) is the beginning of the formulation of the inverse problem. We always regard it as a relation between eigenfunctions analytic in D^+ and those analytic in D^-. Thus, it is necessary for us to introduce the following Riemann–Hilbert problem.

Proposition 8. *Define the sectionally meromorphic matrix*

$$M(x,t,z) = \begin{cases} M^- = \begin{pmatrix} \frac{\mu_{+,1}}{s_{11}} & \mu_{-,2} \end{pmatrix}, \ \text{as } z \in D^-, \\ M^+ = \begin{pmatrix} \mu_{-,1} & \frac{\mu_{+,2}}{s_{22}} \end{pmatrix}, \ \text{as } z \in D^+. \end{cases} \tag{8.1}$$

Then, a multiplicative matrix Riemann–Hilbert problem is proposed:

- *Analyticity: $M(x,t,z)$ is analytic in $\mathbb{C} \setminus \Sigma$ and has single poles.*
- *Jump condition*

$$M^-(x,t,z) = M^+(x,t,z)(I - G(x,t,z)), \qquad z \in \Sigma, \tag{8.2}$$

where

$$G(x,t,z) = \begin{pmatrix} \rho(z)\tilde{\rho}(z) & e^{2i\theta}\tilde{\rho}(z) \\ -e^{-2i\theta}\rho(z) & 0 \end{pmatrix}. \tag{8.3}$$

- *Asymptotic behaviors*

$$M(x,t,z) \sim I + O(z^{-1}), \qquad z \to \infty, \tag{8.4}$$

$$M(x,t,z) \sim -\frac{i}{z}\sigma_3 Q_- + O(1), \qquad z \to 0. \tag{8.5}$$

Proof. The analyticity can be find out from (4.3) and the analyticity of the modified Jost solution $\mu_\pm$.

Also from (4.3), we get

$$\mu_{-,2}(x,t,z) = -\tilde{\rho}(z)e^{2i\theta}\mu_{-,1}(x,t,z) + \frac{\mu_{+,2}(x,t,z)}{s_{22}(z)}, \tag{8.6}$$

$$\begin{aligned} \frac{\mu_{+,1}(x,t,z)}{s_{11}(z)} &= \mu_{-,1}(x,t,z) + \rho(z)\mu_{-,2}(x,t,z) \\ &= (1 - \rho(z)\tilde{\rho}(z))\mu_{-,1}(x,t,z) + \rho(z)e^{-2i\theta}\frac{\mu_{+,2}(x,t,z)}{s_{22}(z)}, \end{aligned} \tag{8.7}$$

which result in the jump condition.

Next we discuss the asymptotic behaviors. With the above asymptotic behaviors of the modified Jost solution and scattering matrix, we can derive that as $z \to \infty$,

$$M^-(x,t,z) \sim I + O(z^{-1}), \tag{8.8}$$

$$M^+(x,t,z) \sim I + O(z^{-1}). \tag{8.9}$$

Thus, we derived the asymptotic behavior (8.4). Similarly, we can get another asymptotic behavior (8.5) immediately. □

Solving the above Riemann–Hilbert problem requires us to regularize it by subtracting out the asymptotic behaviors and the pole contributions. It is convenient to define

$$\zeta_n = \begin{cases} z_n, & n = 1, \ldots, N_1, \\ -\frac{q_0^2}{z^*_{n-N_1}}, & n = N_1 + 1, \ldots, 2N_1, \\ w_{n-2N_1}, & n = 2N_1 + 1, \ldots, 2N_1 + N_2, \end{cases} \tag{8.10}$$

and rewrite (8.2) as

$$\begin{aligned} & M^- - I + \frac{i}{z}\sigma_3 Q_- - \sum_{n=1}^{2N_1+N_2} \frac{\operatorname*{Res}_{\zeta_n^*} M^-}{z-\zeta_n^*} - \sum_{n=1}^{2N_1+N_2} \frac{\operatorname*{Res}_{-\zeta_n^*} M^-}{z+\zeta_n^*} \\ & - \sum_{n=1}^{2N_1+N_2} \frac{\operatorname*{Res}_{\zeta_n} M^+}{z-\zeta_n} - \sum_{n=1}^{2N_1+N_2} \frac{\operatorname*{Res}_{-\zeta_n} M^+}{z+\zeta_n} \\ & = M^+ - I + \frac{i}{z}\sigma_3 Q_- - \sum_{n=1}^{2N_1+N_2} \frac{\operatorname*{Res}_{\zeta_n^*} M^-}{z-\zeta_n^*} - \sum_{n=1}^{2N_1+N_2} \frac{\operatorname*{Res}_{-\zeta_n^*} M^-}{z+\zeta_n^*} \\ & - \sum_{n=1}^{2N_1+N_2} \frac{\operatorname*{Res}_{\zeta_n} M^+}{z-\zeta_n} - \sum_{n=1}^{2N_1+N_2} \frac{\operatorname*{Res}_{-\zeta_n} M^+}{z+\zeta_n} - M^+ G. \end{aligned} \tag{8.11}$$

Apparently, the left-hand side is analytic in D^- and is $O(z^{-1})$ as $z \to \infty$, while the sum of the first five terms of the right-hand side is analytic in D^+ and is $O(z^{-1})$ as $z \to \infty$. In the end, the asymptotic behavior of the off-diagonal scattering coefficients implies that $G(x,t,z)$ is $O(z^{-1})$ as $z \to \infty$ and $O(z)$ as $z \to 0$ along the real axis.

Using Plemelj's formula, we finally get

$$\begin{aligned} M(x,t,z) = {} & I - \frac{i}{z}\sigma_3 Q_- + \sum_{n=1}^{2N_1+N_2} \frac{\operatorname*{Res}_{\zeta_n} M^+}{z-\zeta_n} + \sum_{n=1}^{2N_1+N_2} \frac{\operatorname*{Res}_{\zeta_n^*} M^-}{z-\zeta_n^*} \\ & + \sum_{n=1}^{2N_1+N_2} \frac{\operatorname*{Res}_{-\zeta_n} M^+}{z+\zeta_n} + \sum_{n=1}^{2N_1+N_2} \frac{\operatorname*{Res}_{-\zeta_n^*} M^-}{z+\zeta_n^*} + \frac{1}{2\pi i} \int_\Sigma \frac{M^+(x,t,\zeta)}{\zeta - z} G(x,t,z) \mathrm{d}\zeta, \qquad z \in \mathbb{C} \setminus \Sigma. \end{aligned} \tag{8.12}$$

8.1. Residue conditions and reconstruction formula

Next, we need to derive an expression for the residues which are in (8.12). We know from the residue relations (6.5), (6.10), (6.12), (6.17), (6.28)–(6.31) and (6.34)–(6.37) that only the second column of M^+ has poles at $\pm z_n$, $\pm\frac{q_0^2}{z_n^*}$ and $\pm w_m$. Explicitly that is

$$\operatorname*{Res}_{z=\zeta_n} M^+ = \left(0 \;\, \tilde{C}_n \mathrm{e}^{2i\theta(\zeta_n)} \mu_{-,1}(x,t,\zeta_n)\right), \qquad n = 1, \ldots, 2N_1+N_2, \tag{8.13}$$

$$\operatorname*{Res}_{z=-\zeta_n} M^+ = \left(0 \;\, \tilde{C}_n \mathrm{e}^{2i\theta(\zeta_n)} \sigma_3 \mu_{-,1}(x,t,\zeta_n)\right), \qquad n = 1, \ldots, 2N_1+N_2, \tag{8.14}$$

$$\operatorname*{Res}_{z=\zeta_n^*} M^- = \left(C_n \mathrm{e}^{-2i\theta(\zeta_n^*)} \mu_{-,2}(x,t,\zeta_n^*) \;\, 0\right), \qquad n = 1, \ldots, 2N_1+N_2, \tag{8.15}$$

$$\operatorname*{Res}_{z=-\zeta_n^*} M^- = \left(-C_n \mathrm{e}^{-2i\theta(\zeta_n^*)} \sigma_3 \mu_{-,2}(x,t,\zeta_n^*) \;\, 0\right), \qquad n = 1, \ldots, 2N_1+N_2. \tag{8.16}$$

Hence, we can calculate the second column of M^+ at ζ_n^* and obtain

$$\begin{aligned} \mu_{-,2}(x,t,\zeta_j^*) = {} & \begin{pmatrix} -\frac{i}{\zeta_j^*} q_- \mathrm{e}^{i\nu_-} \\ \mathrm{e}^{-i\nu_-} \end{pmatrix} + 2 \sum_{k=1}^{2N_1+N_2} \frac{\tilde{C}_k \mathrm{e}^{2i\theta(\zeta_k)}}{\zeta_j^{*2} - \zeta_k^2} Z_{jk} \mu_{-,1}(x,t,\zeta_k) \\ & + \frac{1}{2\pi i} \int_\Sigma \frac{M^+(x,t,\zeta)}{\zeta - \zeta_j^*} G(x,t,\zeta) \mathrm{d}\zeta, \end{aligned} \tag{8.17}$$

where $Z_{jk} = \begin{pmatrix} \zeta_j^* & 0 \\ 0 & \zeta_k \end{pmatrix}$, for $j, k = 1, \ldots, 2N_1 + N_2$.

In the same way, we can evaluate the first column of M^- at ζ_n and obtain

$$\begin{aligned} \mu_{-,1}(x,t,\zeta_n) = &\begin{pmatrix} 1 \\ -\frac{i}{\zeta_n} q_-^* \end{pmatrix} + 2 \sum_{j=1}^{2N_1+N_2} \frac{C_j \mathrm{e}^{-2i\theta(\zeta_j^*)}}{\zeta_n^2 - \zeta_j^{*2}} Z_{jn} \mu_{-,2}(x,t,\zeta_j^*) \\ &+ \frac{1}{2\pi i} \int_\Sigma \frac{M^+(x,t,\zeta)}{\zeta - \zeta_n} G(x,t,\zeta) \mathrm{d}\zeta. \end{aligned} \tag{8.18}$$

In the remaining parts of this section, we would like to give the reconstruction formula from the solution of the Riemann–Hilbert problem. The solution (8.12) implies that

$$\begin{aligned} M(x,t,z) = I + \frac{1}{z} \Bigg(&-i\sigma_3 Q_- + \sum_{n=1}^{2N_1+N_2} (\operatorname*{Res}_{z=\zeta_n} M^+ + \operatorname*{Res}_{z=-\zeta_n} M^+ + \operatorname*{Res}_{z=\zeta_n^*} M^- \\ &+ \operatorname*{Res}_{z=-\zeta_n^*} M^-) - \frac{1}{2\pi i} \int_\Sigma M^+(x,t,\zeta) G(x,t,\zeta) \mathrm{d}\zeta \Bigg) + O(z^{-2}), \quad z \to \infty. \end{aligned} \tag{8.19}$$

In the above equation, taking $M = M^-$ and comparing the (1,2) element with the same location of $\frac{\mu_-^{(1)}}{z}$, using (7.8), we get the reconstruction formula for the potential

$$q(x,t) = q_- + 2i \sum_{n=1}^{2N_1+N_2} \tilde{C}_n \mathrm{e}^{2i\theta(\zeta_n)} \mu_{-,11}(\zeta_n) - \frac{1}{2\pi} \int_\Sigma (M^+G)_{1,2}(x,t,\zeta) \mathrm{d}\zeta. \tag{8.20}$$

8.2. Trace formula and theta condition

Recall that the discrete spectrum is composed of $\pm\zeta_n$ and $\pm\zeta_n^*$. Define the functions as follows:

$$\begin{aligned} \beta^+(z) &= s_{22}(z) \prod_{n=1}^{2N_1+N_2} \frac{z^2 - \zeta_n^{*2}}{z^2 - \zeta_n^2}, \\ \beta^-(z) &= s_{11}(z) \prod_{n=1}^{2N_1+N_2} \frac{z^2 - \zeta_n^2}{z^2 - \zeta_n^{*2}}. \end{aligned} \tag{8.21}$$

From the analyticity of the scattering matrix, we see that the above functions are analytic and have no zeros in D^+ and D^-, respectively. When $z \to \infty$, $\beta^\pm(z) \to 1$. Moreover, $\beta^+(z)\beta^-(z) = s_{11}(z)s_{22}(z)$.

Again $\det S(z) = 1$ implies that

$$\frac{1}{s_{11}s_{22}} = \frac{s_{11}s_{22} - s_{12}s_{21}}{s_{11}s_{22}} = 1 - \rho(z)\tilde{\rho}(z) = 1 + \rho(z)\rho^*(z^*); \tag{8.22}$$

thus,

$$\beta^+(z)\beta^-(z) = \frac{1}{1 + \rho(z)\rho^*(z^*)}, \qquad z \in \Sigma. \tag{8.23}$$

Taking logarithms to the above relation and using the Plemelj' formula, we have

$$\log \beta_\pm(z) = \mp \frac{1}{2\pi i} \int_\Sigma \frac{\log[1 + \rho(\zeta)\rho^*(\zeta^*)]}{\zeta - z} \mathrm{d}\zeta, \qquad z \in D^\pm. \tag{8.24}$$

Substituting $\beta^+(z)$ for $s_{11}(z)$, we then obtain

$$s_{11}(z) = \exp\left[-\frac{1}{2\pi i}\int\limits_{\Sigma} \frac{\log[1+\rho(\zeta)\rho^*(\zeta^*)]}{\zeta - z}\mathrm{d}\zeta\right] \prod_{n=1}^{2N_1+N_2} \frac{z^2-\zeta_n^{*2}}{z^2-\zeta_n^2}, \qquad z\in D^-, \tag{8.25}$$

which is called trace formula and express the analytic scattering coefficient in terms of the discrete eigenvalues and the reflection coefficient. In the same way, we can obtain $s_{22}(z)$

$$s_{22}(z) = \exp\left[\frac{1}{2\pi i}\int\limits_{\Sigma} \frac{\log[1+\rho(\zeta)\rho^*(\zeta^*)]}{\zeta - z}\mathrm{d}\zeta\right] \prod_{n=1}^{2N_1+N_2} \frac{z^2-\zeta_n^{2}}{z^2-\zeta_n^{*2}}, \qquad z\in D^+. \tag{8.26}$$

Recalling the asymptotic behavior of the scattering matrix and taking the limit $z\to 0$ of (8.25), we then obtain the so-called theta condition

$$\arg\left(\frac{q_-}{q_+}\right) = 16\sum_{n=1}^{N_1}\arg(z_n) + 8\sum_{m=1}^{N_2}\arg(w_m) + \frac{1}{2\pi}\int\limits_{\Sigma}\frac{\log[1+\rho(\zeta)\rho^*(\zeta^*)]}{\zeta}\mathrm{d}\zeta. \tag{8.27}$$

9. Multi-soliton solutions

9.1. The formula for N-soliton solutions

Now we focus on the potentials $q(x,t)$ with the reflection coefficient $\rho(z)=0$. For convenience, denote

$$c_j(x,t,z) = \frac{C_j}{z^2-\zeta_j^{*2}}\mathrm{e}^{-2i\theta(x,t,\zeta_j^*)}, \qquad j=1,\ldots,2N_1+N_2. \tag{9.1}$$

Observing (8.20), we find that only the first component of the eigenfunction is required in the reconstruction formula. So that we can derive

$$\mu_{-,12}(x,t,\zeta_j^*) = -\frac{i}{\zeta_j^*}q_- - \sum_{k=1}^{2N_1+N_2} 2\zeta_j^* c_k^*(\zeta_j)\mu_{-,11}(x,t,\zeta_k), \qquad j=1,\ldots,2N_1+N_2, \tag{9.2}$$

$$\mu_{-,11}(x,t,\zeta_n) = 1 + \sum_{j=1}^{2N_1+N_2} 2\zeta_j^* c_j(\zeta_n)\mu_{-,12}(x,t,\zeta_j^*), \qquad n=1,\ldots,2N_1+N_2. \tag{9.3}$$

Substituting (9.2) in (9.3), we get

$$\mu_{-,11}(x,t,\zeta_n) = 1 - 2iq_-\sum_{j=1}^{2N_1+N_2} c_j(\zeta_n) - \sum_{j=1}^{2N_1+N_2}\sum_{k=1}^{2N_1+N_2} 4\zeta_j^{*2}c_j(\zeta_n)c_k^*(\zeta_j)\mu_{-,11}(x,t,\zeta_k), \tag{9.4}$$

for $n=1,\ldots,2N_1+N_2$.

Next, to get the brief expression of the reflectionless potentials, we would like to write this system in matrix form. Let

$$\mathbf{X} = (X_1,\ldots,X_{2N_1+N_2})^t, \qquad \mathbf{B} = (B_1,\ldots,B_{2N_1+N_2})^t,$$

where

$$X_n = \mu_{-,11}(x,t,\zeta_n), \qquad B_n = 1-2iq_-\sum_{j=1}^{2N_1+N_2} c_j(\zeta_n), \qquad n=1,\ldots,2N_1+N_2.$$

Define the $(2N_1+N_2)\times(2N_1+N_2)$ matrix $A=(A_{nk})$, where

$$A_{nk} = \sum_{j=1}^{2N_1+N_2} 4\zeta_j^{*2}c_j(\zeta_n)c_k^*(\zeta_j), \qquad n,k=1,\ldots,2N_1+N_2.$$

Then the system (9.4) becomes

$$P\mathbf{X} = \mathbf{B},$$

where $P = I + A = (\mathbf{P}_1, \ldots, \mathbf{P}_{2N_1+N_2})$. The solution of the system is

$$X_n = \frac{\det P_n^{\mathrm{ext}}}{\det P}, \qquad n = 1, \ldots, 2N_1 + N_2, \tag{9.5}$$

where

$$P_n^{\mathrm{ext}} = (\mathbf{P}_1, \ldots, \mathbf{P}_{n-1}, \mathbf{B}, \mathbf{P}_{n+1}, \ldots, \mathbf{P}_{2N_1+N_2}).$$

Therefore, putting the above $X_1, \ldots, X_{2N_1+N_2}$ into the reconstruction formula, we then get the brief expression for the potential

$$q(x,t) = q_- - 2i\frac{\det P^{\mathrm{aug}}}{\det P}, \tag{9.6}$$

where the $(2N_1 + N_2 + 1) \times (2N_1 + N_2 + 1)$ matrix is given as

$$P^{\mathrm{aug}} = \begin{pmatrix} 0 & \mathbf{Y}^t \\ \mathbf{B} & M \end{pmatrix}, \qquad \mathbf{Y} = (Y_1, \ldots, Y_{2N_1+N_2})^t,$$

and

$$Y_n = \tilde{C}_n \mathrm{e}^{2i\theta(x,t,\zeta_n)}, \qquad n = 1, \ldots, 2N_1 + N_2.$$

As applications of the N-soliton formula (9.6), we would like to consider the one-soliton solutions and two-soliton solutions. Recall that the GI equation is invariant under scaling: If $q(x,t)$ solves the equation, so dose $cq(x,t)$, for $c \in \mathbb{R}$. This allows us to restrict ourselves to the case $q_0 = 1$ without loss of generality.

9.2. One-soliton solutions for $N = 1$

▶ One-breather with parameters $N_1 = 1$ and $N_2 = 0$.

Let $\zeta_1 = Z\mathrm{e}^{i\alpha}$, with $Z > 1$ and $\alpha \in (0, \frac{\pi}{2})$, then the other points in discrete spectrum are $\zeta_2 = -\frac{1}{Z}\mathrm{e}^{i\alpha}$, $-\zeta_1 = -Z\mathrm{e}^{i\alpha}$, $-\zeta_2 = \frac{1}{Z}\mathrm{e}^{i\alpha}$, $\zeta_1^* = Z\mathrm{e}^{-i\alpha}$, $\zeta_2^* = -\frac{1}{Z}\mathrm{e}^{-i\alpha}$, $-\zeta_1^* = -Z\mathrm{e}^{-i\alpha}$ and $-\zeta_2^* = \frac{1}{Z}\mathrm{e}^{-i\alpha}$. By using the theta condition (8.27), we have

$$\arg(q_-/q_+) = 16\alpha.$$

We set $q_- = 1$ and $q_+ = \mathrm{e}^{-16i\alpha}$. And we can also know that $C_1 = \mathrm{e}^{\xi + i\varphi}$, with ξ, $\varphi \in \mathbb{R}$ and $C_2 = -\frac{1}{Z^2}\mathrm{e}^{\xi - i(2\alpha+\varphi)}$.

Substituting above data in formula (9.6), we get the one-soliton solution

$$q(x,t) = 1 - 2i\frac{\det\begin{pmatrix} 0 & Y_1 & Y_2 \\ B_1 & 1 + A_{11} & A_{12} \\ B_2 & A_{21} & 1 + A_{22} \end{pmatrix}}{\det\begin{pmatrix} 1 + A_{11} & A_{12} \\ A_{21} & 1 + A_{22} \end{pmatrix}}, \tag{9.7}$$

where

$$\theta(x,t,\zeta_j) = -\frac{1}{4}(\zeta_j^2 - \frac{1}{\zeta_j^2})(x + (\frac{1}{2}(\zeta_j^2 + \frac{1}{\zeta_j^2}) - 2)t), \qquad j = 1, 2,$$

$$c_j(x,t,z) = \frac{C_j}{z^2 - \zeta_j^{*2}}\mathrm{e}^{-2i\theta(x,t,\zeta_j^*)}, \qquad j = 1, 2,$$

$$B_n = 1 - 2iq_- \sum_{j=1}^{2} c_j(\zeta_n), \qquad n = 1, 2,$$

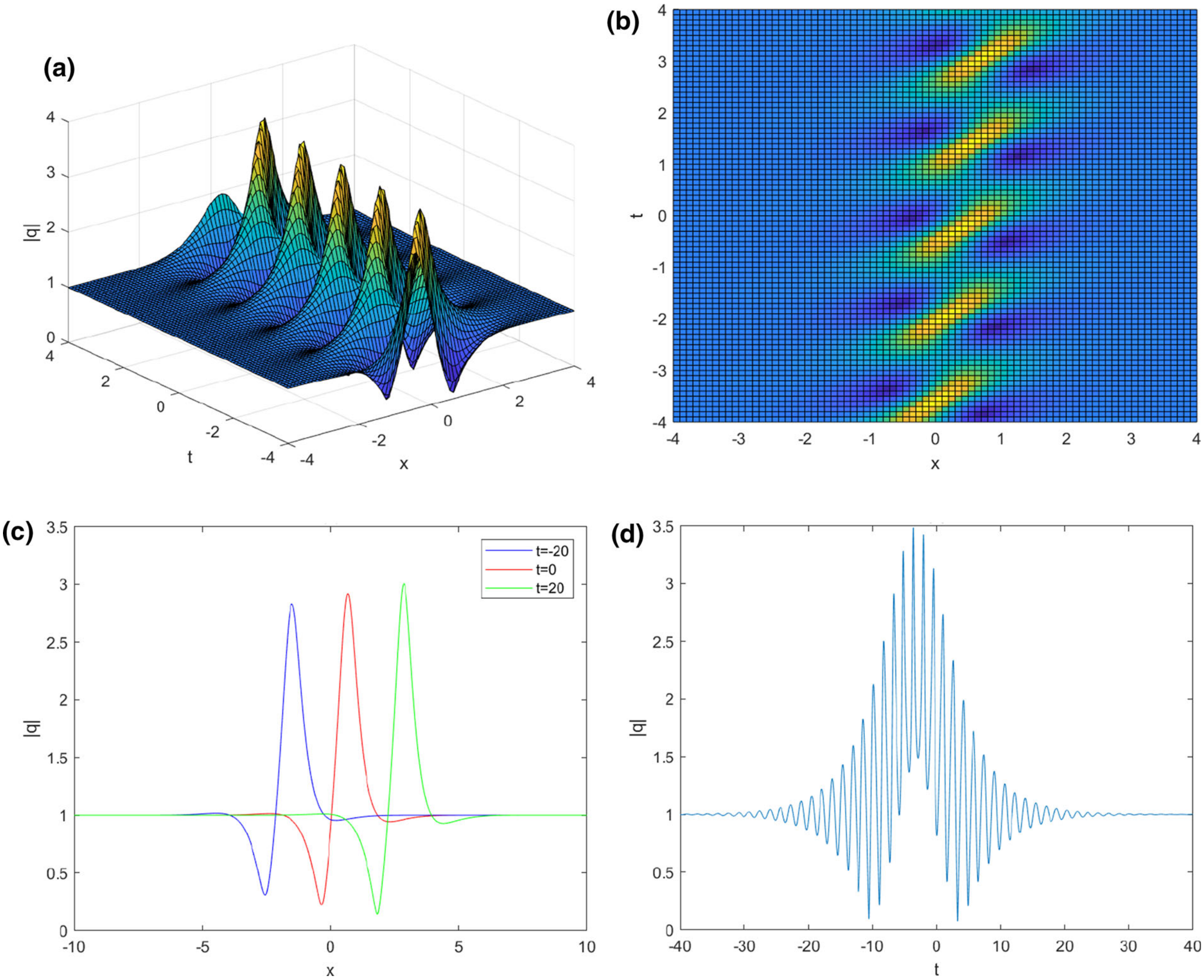

FIG. 3. One-breather with parameters $N_1 = 1$, $N_2 = 0$, $z_1 = 2e^{\frac{\pi}{6}i}$, $C_1 = i$. **a** The three-dimensional graph of one-soliton solution. **b** The contour of the wave. **c** Wave propagations along $x-$orientation with $t = -20, 0$ and $t = 20$. **d** Wave propagation along $t-$orientation with $x = 0$

$$A_{nk} = \sum_{j=1}^{2} 4\zeta_j^{*2} c_j(\zeta_n) c_k^*(\zeta_j), \qquad n, k = 1, 2,$$
$$Y_n = -C_n^* e^{2i\theta(x,t,\zeta_n)}, \qquad n = 1, 2.$$

The properties of the one-soliton solution are shown in Fig. 3.

► One-single soliton with parameters $N_1 = 0$ and $N_2 = 1$.

Let $\zeta_1 = e^{i\beta}$, with $\beta \in (0, \frac{\pi}{2})$, then the discrete spectrum can be expressed as $\{e^{i\beta}, -e^{i\beta}, e^{-i\beta}, -e^{-i\beta}\}$. By using theta condition (8.27), we get

$$\arg(q_-/q_+) = 8\beta. \tag{9.8}$$

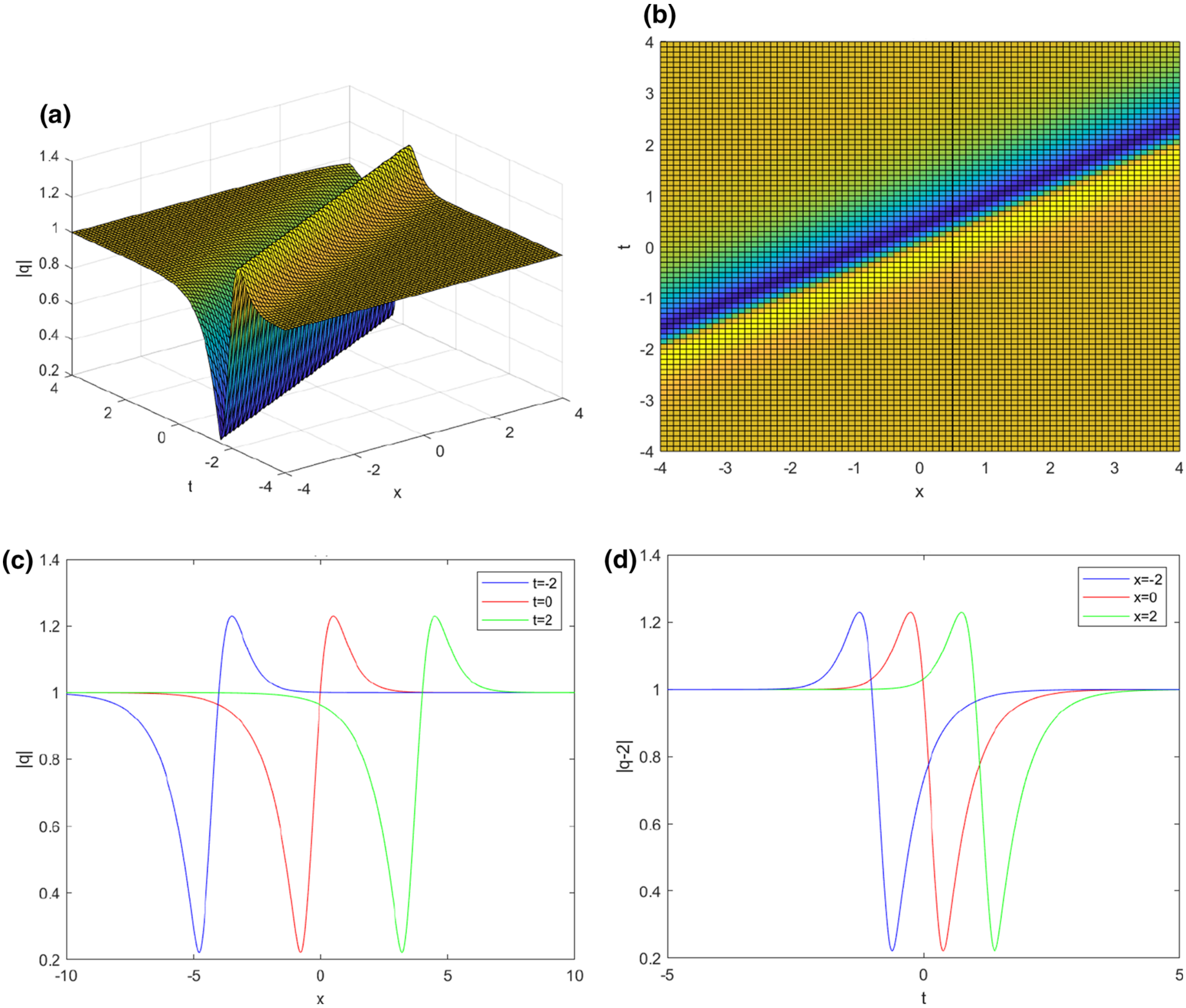

FIG. 4. One-single soliton with parameters $N_1 = 0$, $N_2 = 1$, $w_1 = \mathrm{e}^{\frac{\pi}{4}i}$, $C_1 = i$. **a** The three-dimensional graph of one-soliton solution. **b** The contour of the wave. **c** Wave propagations along $x-$orientation with $t = -2, 0$ and $t = 2$. **d** Wave propagation along $t-$orientation with $x = -2, 0$ and $t = 2$

We set $q_- = 1$ and then $q_+ = \mathrm{e}^{-8i\beta}$. Let $C_1 = \mathrm{e}^{i\tau+\kappa}$, with $\tau, \kappa \in \mathbb{R}$. Once again we get the soliton solution with parameters $N_1 = 0$ and $N_2 = 1$

$$q(x,t) = 1 - 2i\frac{\det\begin{pmatrix} 0 & Y \\ B & 1+A \end{pmatrix}}{1+A}, \tag{9.9}$$

where

$$\theta(x,t,\zeta_1) = -\frac{1}{2}\sinh(2i\beta)\big(x + (\cosh(2i\beta) - 2)t\big),$$
$$c_1(x,t,\zeta_1) = \frac{\mathrm{e}^{i\tau+\kappa}}{\sinh(2i\beta)}\mathrm{e}^{-2i\theta(x,t,\zeta_1^*)},$$

$$B = 1 - 2ic_1(x,t,\zeta_1),$$
$$A = 4\zeta_1^{*2} c_1(\zeta_1) c_1^*(\zeta_1),$$
$$Y = -C_1^* \mathrm{e}^{2i\theta(x,t,\zeta_1)}.$$

The properties of the one-soliton solution are shown in Fig. 4.

9.3. Two-soliton solutions for $N = 2$

▶ Two-breather with $N_1 = 1$ and $N_2 = 1$.

Let $\zeta_1 = Z\mathrm{e}^{i\alpha}$ with $Z > 1$ and $\alpha \in (0, \frac{\pi}{2})$. Let $\zeta_3 = \mathrm{e}^{i\beta}$, with $\beta \in (0, \frac{\pi}{2})$. By using the theta condition (8.27), we have

$$\arg(q_-/q_+) = 16\alpha + 8\beta.$$

We set $q_- = 1$ and $q_+ = \mathrm{e}^{-8i(2\alpha+\beta)}$. And we can also know that $C_1 = \mathrm{e}^{\xi + i\varphi}$, $C_3 = \mathrm{e}^{i\tau+\kappa}$, with $\tau, \kappa, \xi, \varphi \in \mathbb{R}$, then $C_2 = -\frac{1}{Z^2}\mathrm{e}^{\xi - i(2\alpha+\varphi)}$. Substituting above data in formula (9.6), we get the two-soliton solution

$$q(x,t) = 1 - 2i \frac{\det \begin{pmatrix} 0 & Y_1 & Y_2 & Y_3 \\ B_1 & 1 + A_{11} & A_{12} & A_{13} \\ B_2 & A_{21} & 1 + A_{22} & A_{23} \\ B_3 & A_{31} & A_{32} & A_{33} + 1 \end{pmatrix}}{\det \begin{pmatrix} 1 + A_{11} & A_{12} & A_{13} \\ A_{21} & 1 + A_{22} & A_{23} \\ A_{31} & A_{32} & 1 + A_{33} \end{pmatrix}}, \tag{9.10}$$

where

$$\theta(x,t,\zeta_j) = -\frac{1}{4}(\zeta_j^2 - \frac{1}{\zeta_j^2})(x + (\frac{1}{2}(\zeta_j^2 + \frac{1}{\zeta_j^2}) - 2)t), \qquad j = 1,2,3,$$
$$c_j(x,t,z) = \frac{C_j}{z^2 - \zeta_j^{*2}} \mathrm{e}^{-2i\theta(x,t,\zeta_j^*)}, \qquad j = 1,2,3,$$
$$B_n = 1 - 2iq_- \sum_{j=1}^{3} c_j(\zeta_n), \qquad n = 1,2,3,$$
$$A_{nk} = \sum_{j=1}^{3} 4\zeta_j^{*2} c_j(\zeta_n) c_k^*(\zeta_j), \qquad n,k = 1,2,3,$$
$$Y_n = -C_n^* \mathrm{e}^{2i\theta(x,t,\zeta_n)}, \qquad n = 1,2,3.$$

The properties of the two-soliton solution are shown in Fig. 5.

▶ Two-soliton with $N_1 = 2$ and $N_2 = 0$.

Let $\zeta_j = Z_j \mathrm{e}^{i\alpha_j}$, with $j = 1,2$, $Z_j > 1$ and $\alpha_j \in (0, \frac{\pi}{2})$. By using the theta condition (8.27), we have

$$\arg(q_-/q_+) = 16(\alpha_1 + \alpha_2).$$

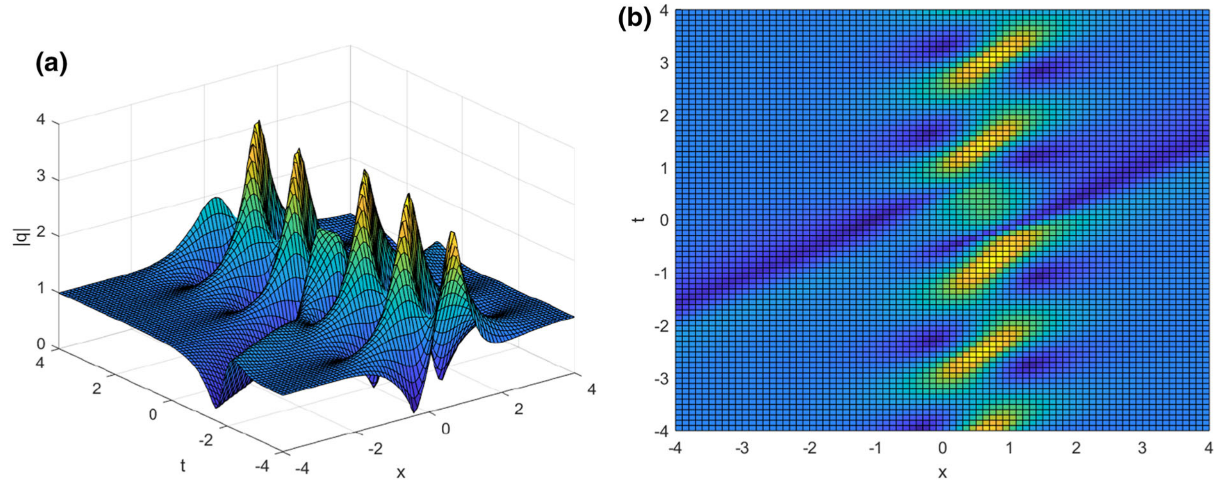

FIG. 5. Two-breather with parameters $N_1 = 1$, $N_2 = 1$, $w_1 = \mathrm{e}^{\frac{\pi}{4}i}$, $z_1 = 2\mathrm{e}^{\frac{\pi}{6}i}$, $C_1 = C_2 = C_3 = i$. **a** The three-dimensional graph. **b** Then contour of the wave. **c** Wave propagations along $x-$orientation with $t = 0$. **d** Wave propagation along $t-$orientation with $x = 0$

We set $q_- = 1$ and $q_+ = \mathrm{e}^{-16i(\alpha_1+\alpha_2)}$. And we can also know that $C_1 = \mathrm{e}^{\xi+i\varphi}$, $C_2 = \mathrm{e}^{\tau+i\kappa}$ with $\xi, \varphi, \tau, \kappa \in \mathbb{R}$. Substituting above data in formula (9.6), we get the two-soliton solution

$$q(x,t) = 1 - 2i\frac{\det\begin{pmatrix} 0 & Y_1 & Y_2 & Y_3 & Y_4 \\ B_1 & 1+A_{11} & A_{12} & A_{13} & A_{14} \\ B_2 & A_{21} & 1+A_{22} & A_{23} & A_{24} \\ B_3 & A_{31} & A_{32} & 1+A_{33} & A_{34} \\ B_4 & A_{41} & A_{42} & A_{43} & 1+A_{44} \end{pmatrix}}{\det\begin{pmatrix} 1+A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & 1+A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & 1+A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & 1+A_{44} \end{pmatrix}}, \tag{9.11}$$

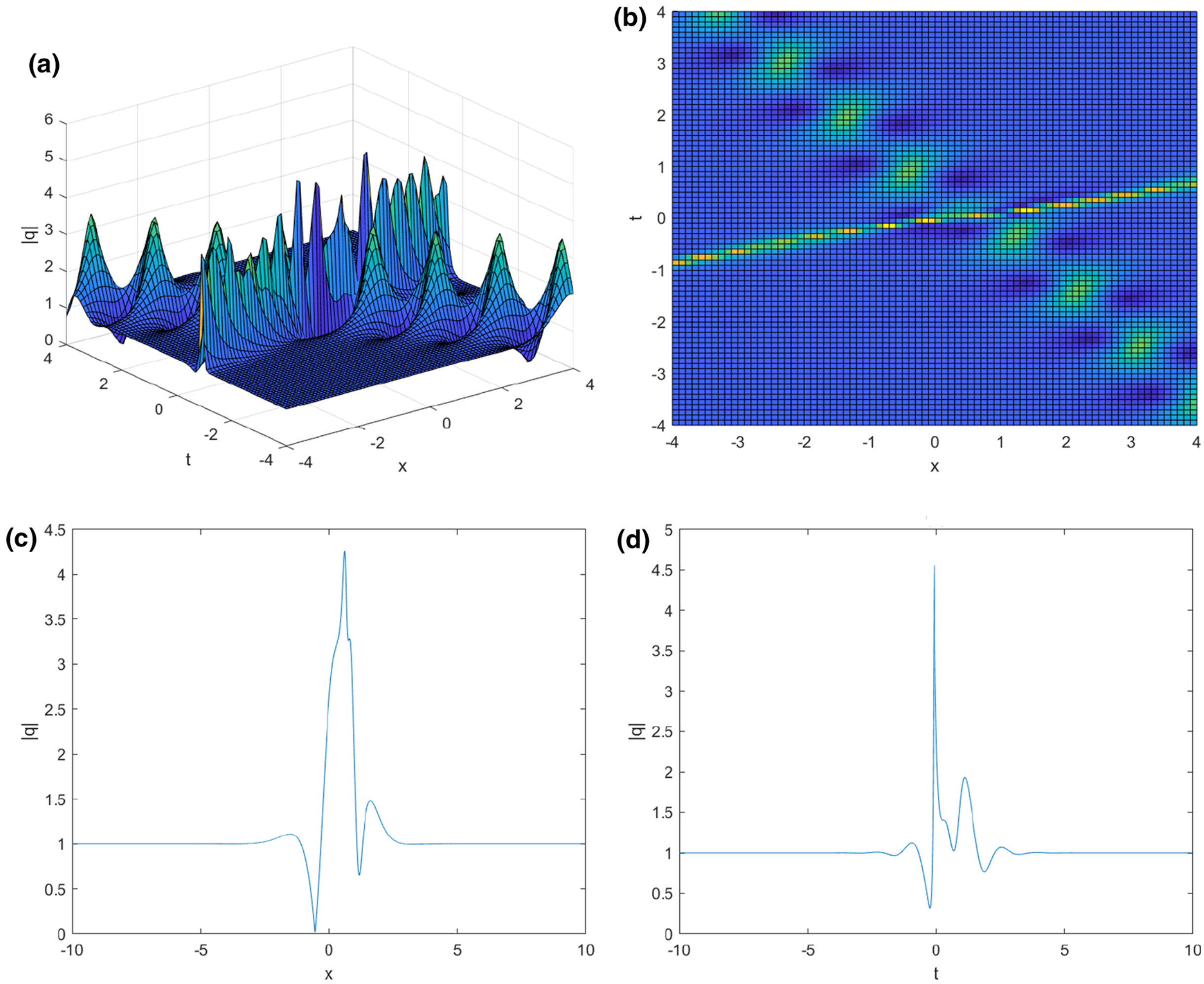

Fig. 6. Two-soliton with parameters $N_1 = 2$, $N_2 = 0$, $z_1 = 1 + 2i$, $z_2 = 2 + i$, $C_1 = C_2 = i$. **a** The three-dimensional graph. **b** The contour of the wave. **c** Wave propagations along $x-$orientation with $t = 0$. **d** Wave propagation along $t-$orientation with $x = 0$

where

$$\theta(x,t,\zeta_j) = -\frac{1}{4}(\zeta_j^2 - \frac{1}{\zeta_j^2})(x + (\frac{1}{2}(\zeta_j^2 + \frac{1}{\zeta_j^2}) - 2)t), \qquad j = 1,2,3,4,$$

$$c_j(x,t,z) = \frac{C_j}{z^2 - \zeta_j^{*2}} \mathrm{e}^{-2i\theta(x,t,\zeta_j^*)}, \qquad j = 1,2,3,4,$$

$$B_n = 1 - 2iq_- \sum_{j=1}^{4} c_j(\zeta_n), \qquad n = 1,2,3,4,$$

$$A_{nk} = \sum_{j=1}^{4} 4\zeta_j^{*2} c_j(\zeta_n) c_k^*(\zeta_j), \qquad n,k = 1,2,3,4,$$

$$Y_n = -C_n^* \mathrm{e}^{2i\theta(x,t,\zeta_n)}, \qquad n = 1,2,3,4.$$

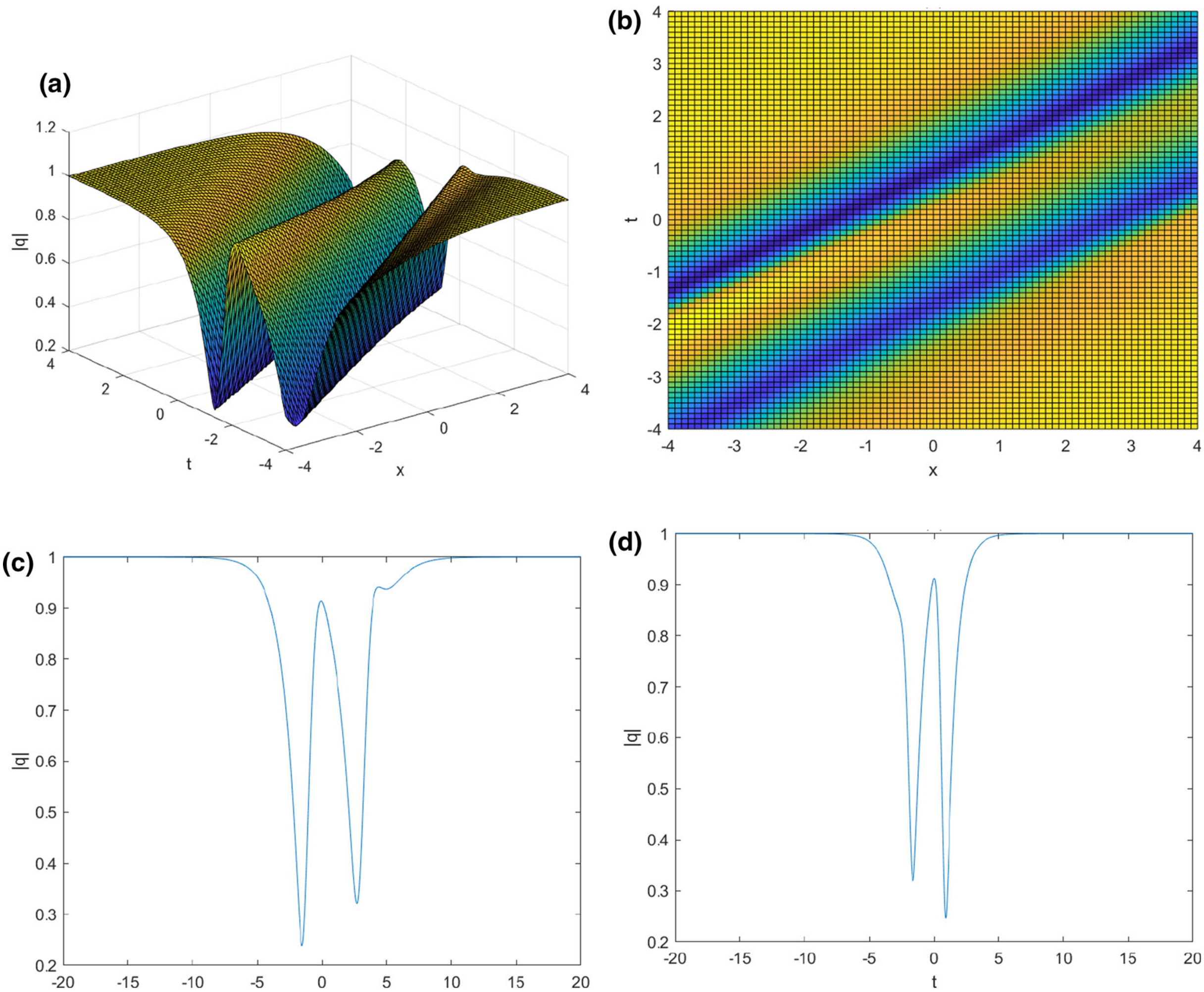

FIG. 7. Two-soliton with parameters $N_1 = 0$, $N_2 = 2$, $w_1 = \mathrm{e}^{\frac{\pi i}{6}}$, $w_2 = \mathrm{e}^{\frac{\pi i}{4}}$, $C_1 = C_2 = i$. **a** The three-dimensional graph. **b** The contour of the wave. **c** Wave propagations along $x-$orientation with $t = 0$. **d** Wave propagation along $t-$orientation with $x = 0$

The properties of the one-soliton solution are shown in Fig. 6.

► Two-soliton solution with $N_1 = 0$ and $N_2 = 2$.

Let $\zeta_j = \mathrm{e}^{i\beta_j}$, with $\beta_j \in (0, \frac{\pi}{2})$ and $j = 1, 2$. By using theta condition (8.27), we get

$$\arg(q_-/q_+) = 8(\beta_1 + \beta_2). \tag{9.12}$$

We set $q_- = 1$ and then $q_+ = \mathrm{e}^{-8i(\beta_1+\beta_2)}$. Let $C_1 = \mathrm{e}^{i\tau+\kappa}$ and $C_2 = \mathrm{e}^{\xi+i\varphi}$, with $\tau, \kappa, \xi, \varphi \in \mathbb{R}$. Once again we get the two-soliton solution with parameters $N_1 = 0$ and $N_2 = 2$

$$q(x,t) = 1 - 2i \frac{\det \begin{pmatrix} 0 & Y_1 & Y_2 \\ B_1 & 1 + A_{11} & A_{12} \\ B_2 & A_{21} & 1 + A_{22} \end{pmatrix}}{\det \begin{pmatrix} 1 + A_{11} & A_{12} \\ A_{21} & 1 + A_{22} \end{pmatrix}}, \tag{9.13}$$

where

$$
\begin{aligned}
&\theta(x,t,\zeta_j) = -\frac{1}{4}(\zeta_j^2 - \frac{1}{\zeta_j^2})(x + (\frac{1}{2}(\zeta_j^2 + \frac{1}{\zeta_j^2}) - 2)t), \qquad j = 1, 2,\\
&c_j(x,t,z) = \frac{C_j}{z^2 - \zeta_j^{*2}} \mathrm{e}^{-2i\theta(x,t,\zeta_j^*)}, \qquad j = 1, 2,\\
&B_n = 1 - 2iq_- \sum_{j=1}^{2} c_j(\zeta_n), \qquad n = 1, 2,\\
&A_{nk} = \sum_{j=1}^{2} 4\zeta_j^{*2} c_j(\zeta_n) c_k^*(\zeta_j), \qquad n, k = 1, 2,\\
&Y_n = -C_n^* \mathrm{e}^{2i\theta(x,t,\zeta_n)}, \qquad n = 1, 2.
\end{aligned}
$$

The properties of the two-soliton solution are shown in Fig. 7.

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