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Soliton resolution and asymptotic stability of N-soliton solutions for the defocusing mKdV equation with a non-vanishing background

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ABSTRACT

We analytically study the large-time asymptotics of the solution of the defocusing modified Korteweg–de Vries (mKdV) equation under a symmetric non-vanishing background, which supports the emergence of solitons. It is demonstrated that the asymptotic expansion of the solution at the large time could verify the renowned soliton resolution conjecture. Moreover, the asymptotic stability of N-soliton solution is also exhibited in the present work. We establish our results by performing a $\bar{\partial}$-nonlinear steepest descent analysis to the associated Riemann–Hilbert (RH) problem.

1. Introduction

We investigate the Cauchy problem for the defocusing modified Korteweg–de Vries (mKdV) equation with finite density initial data

$$q_t(x,t) + q_{xxx}(x,t) - 6q^2(x,t)q_x(x,t) = 0, \quad (x,t) \in \mathbb{R}\times\mathbb{R}^+, \tag{1.1}$$

$$q(x,0) = q_0(x) \to \pm 1, \quad x \to \pm\infty. \tag{1.2}$$

Let $\lim_{x\to\pm\infty} q_0(x) = q_\pm$. Note that, if $q_\pm = \pm A$ as $x \to \pm\infty$, one can always reduce oneself to either $A = 1$ or $A = -1$ without loss of generality thanks to scaling invariance of mKdV equation. Indeed, one can make $u = A^{-1}q$, $\tilde{x} = Ax$ and $\tilde{t} = A^3 t$, it is consequent that $u(\tilde{x},\tilde{t})$ satisfies (1.1) with the normalized boundary condition $u_0(\tilde{x}) \to \pm 1$ as $\tilde{x} \to \pm\infty$. Note also that the case $q_\pm = \mp 1$ is trivially reduced to the present one thanks to the invariance of the mKdV equation under change of sign [i.e., the transformation $q(x,t) \mapsto -u(x,t)$].

The mKdV equation arises in various physical fields, such as acoustic wave and phonons in a certain anharmonic lattice [1,2], Alfén wave in a cold collision-free plasma [3], meandering ocean currents [4], hyperbolic surfaces [5], and Schottky barrier transmission [6]. There are plenty of results on the mathematical properties for the mKdV equation. Here we cite only those that are closed to our study. In the late 1970s, the inverse scattering theory was applied to solve the mKdV equation and investigate large-time asymptotics for the mKdV equation. For example, Wadati investigated the focusing mKdV equation with zero boundary conditions and derived simple-pole, double-pole and triple-pole solutions [7,8]. The long-time behavior of the defocusing mKdV equation with given Schwartz initial data is provided by Segur and Ablowitz without consideration of solitons [9]. Deift and Zhou developed nonlinear steepest descent method and obtained the long-time asymptotic behavior of the defocusing mKdV equation with the Schwartz initial data in their seminal work [10]. This approach was further developed into a $\bar{\partial}$ steepest descent method by McLaughlin and Miller to analyze asymptotics of orthogonal polynomials with non-analytical weights [11,12]. Later, with Dieng, they applied it to investigate the defocusing NLS equation under essentially minimal regularity assumptions on finite mass initial data [13]. Boutet de Monvel et al. discussed the initial boundary value problem of defocusing mKdV equation on the half line by using the Fokas method [14]. For the weighted Sobolev initial data, Chen and Liu et al. have studied the large-time asymptotic behavior of defocusing mKdV equation with zero boundary conditions without consideration of solitons [15], and the long-time asymptotic behavior of focusing mKdV equation with

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Fig. 1. The (x,t)-plane is divided into three kinds of asymptotic regions: Solitonic region, $-6<\xi\le -2$; Solitonless region, $\xi<-6$ and $\xi>-2$; Transition region, $\xi\approx -6$. Here, $\xi:=x/t$.

zero boundary conditions with solitons [16]. However, for defocusing mKdV equation with nonzero boundary conditions, soliton solutions will appear due to non-empty discrete spectrum for finite mass initial data. It is necessary to consider the effect of soliton solutions when we study large-time asymptotic behavior, which naturally require a more detailed necessary description to obtain the large-time asymptotics of the defocusing mKdV equation.

Our goal in this paper is to give detailed asymptotic analysis for the defocusing mKdV equation (1.1) with finite density type initial data in the given space–time solitonic regions $|x/t+4|<2$; see Fig. 1 for an illustration.

We investigate the asymptotic stability and soliton resolution for the mKdV equation (1.1) for the region $|x/t+4|<2$, in which there are no phase points on the real axis. For the case of solitonless regions $|x/t+4|>2$, it is considered in [17]. Based on the phase velocity ξ, apart from the critical line at $\xi=-6$, the other critical line should technically be at $\xi=6$ (refer the Proposition 2.10). However, for the region where x/t belongs to $[-2,6)$, we find that the set of solitons contributing to the asymptotic behavior is empty. This indicates that the region $[-2,6)$ can be seen as a special case of the solitonic region by setting the index Λ defined in (3.2) be empty, and can also be classified as a solitonless region. Here, we consider the region $[-2,6)$ as part of the solitonless region, which leads to the classification shown in Fig. 1. The only transient region in Fig. 1 is near $\xi=-6$, which we have made a discussion in [18].

The soliton resolution conjecture is one of the most interesting phenomenon observed in the study of solutions to certain nonlinear dispersive partial differential equations (PDEs). The conjecture suggests that solutions with generic initial data for many dispersive equations should eventually decompose into a finite number of solitons, each moving at different speeds, along with a radiative term [19–24]. Understanding and proving soliton resolution contribute to our broader understanding of the behavior of nonlinear dispersive systems, shedding light on the intricate interplay between nonlinearities, dispersion, and soliton dynamics. Recently, large-time asymptotics and soliton resolution for some integrable systems have been obtained by using $\bar\partial$-generalization of the nonlinear steepest descent method [25–30].

This paper is organized as follows. In Section 2, we get down to the spectral analysis on the Lax pair. We state the symmetries, asymptotic behaviors and time evolution of the scattering data. Further discussion show that the zeros of $a(z)$ are simple and finite. In Section 3, we set up an RH problem for a sectionally meromorphic function $m(z)$ comprised by the Jost solutions and the scattering data. Once the solution of the RH problem exists, we can directly obtain the reconstruction formula. To handle the RH problem , we first give the distributions of phase points and the signature table of $\mathrm{Re}(2it\theta)$, then introduce a set of conjugations and interpolations, such that the original RH problem becomes a standard RH problem. Subsequently, according to the basic factorization of the jump matrix, we introduce some appropriate extensions to deform the jumps onto four different contours in the complex plane on which their forms are asymptotically small. In Section 4, by neglecting the $\bar\partial$ term of $m^{(2)}(z)$, we get a conjugation of the RH problem related to the N-soliton with the modified scattering data. In this way, we can consider the asymptotic behavior of N-soliton solutions by using a small norm theorem. The existence of $m^{(3)}(z)$ is verified, as well as its asymptotic estimate. Finally, in Section 5, we give the proofs of the main theorems applying the above consequences.

Notation. We first provide some notations used in this paper:

- $\mathbb{R}^+=(0,\infty)$, $\mathbb{C}^\pm=\{z\in\mathbb{C}:\pm\mathrm{Im}\,z>0\}$.
- The Japanese bracket is defined as $\langle x\rangle:=\sqrt{1+|x|^2}$.
- The normed space $L^{p,s}(\mathbb{R})$ is defined with $\|q\|_{L^{p,s}(\mathbb{R})}\doteq\|\langle x\rangle^s q\|_{L^p(\mathbb{R})}$; $W^{k,p}(\mathbb{R})$ is defined with $\|q\|_{W^{k,p}(\mathbb{R})}\doteq\sum_{j=0}^k\|\partial^j q\|_{L^p(\mathbb{R})}$; $H^k(\mathbb{R})$ is defined with $\|q\|_{H^k(\mathbb{R})}\doteq\|\langle x\rangle^k\hat q\|_{L^2(\mathbb{R})}$, where $\hat u$ is the Fourier transform of u, and $H^{k,k}(\mathbb{R})\doteq L^{2,k}(\mathbb{R})\cap H^k(\mathbb{R})$.
- $\sigma_i(i=1,2,3)$ are the Pauli matrices defined as

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{pmatrix}\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{1.3}$$

Main results. The main results of the work are listed below.

Theorem 1.1. *Suppose the initial data $q_0\mp 1\in H^{4,4}(\mathbb{R}^\pm)$ with scattering data $\left\{r(z),\{z_j,c_j\}_{j=0}^{N-1}\right\}$. Order z_j such that*

$$\mathrm{Re}\,z_0>\mathrm{Re}\,z_1>\cdots>\mathrm{Re}\,z_{N-1}\ge 0, \tag{1.4}$$

and define $\xi=\frac{x}{t}$. Let $q^{(sol),N}(x,t)$ be the N-soliton solution whose scattering data can be denoted by $\{\tilde r\equiv 0,\{z_j,\tilde c_j\}_{j=0}^{N-1}\}$, where

$$\tilde c_j=c_j\exp\left(-\frac{1}{i\pi}\int_{\mathbb{R}}\log(1-|r(s)|^2)\left(\frac{1}{s-z_j}-\frac{1}{2s}\right)ds\right). \tag{1.5}$$

For fixed $\xi_0\in(0,2)$, there exist constants $t_0=t_0(q_0,\xi_0)$ and $C=C(q_0,\xi_0)$ such that the potential $q(x,t)$ of (2.1) satisfies

$$|q(x,t)-q^{(sol),N}(x,t)|\le ct^{-1},\qquad t>t_0,\quad |\xi+4|\le\xi_0. \tag{1.6}$$

Furthermore, for $t>t_0$ and $|\xi+4|<\xi_0$, we have confirmed the soliton resolution for the N-soliton solution

$$q(x,t)=-1+\sum_{j=0}^{N-1}[sol(z_j;x-x_j,t)+1]+\mathcal{O}(t^{-1}), \tag{1.7}$$

where $sol(z_j;x-x_j,t)$ is defined by (4.10), and

$$x_j=\frac{1}{2\,\mathrm{Im}\,z_j}\left\{\log\left(\frac{|c_j|}{\mathrm{Im}\,z_j}\prod_{k\in\Delta,k\ne j}\left|\frac{(z_j-z_k)(z_j+\bar z_k)}{(z_jz_k-1)(z_j\bar z_k+1)}\right|\right).\right.$$
$$\left.-\frac{\mathrm{Im}\,z_j}{\pi}\int_{\mathbb{R}}\frac{\log(1-|r(s)|^2)}{|s-z_j|^2}ds\right\}.$$

As a corollary of Theorem 1.1, we have the following theorem:

Theorem 1.2. *Let $q^{(sol),M}(x,t)$ be an M-soliton satisfying the boundary conditions in (1.2). Let $\left\{0,\{z_j,c_j\}_{j=0}^{M-1}\right\}$ denote its reflectionless scattering data. Then there exist ε_0 and $C>0$, for any initial data q_0 of problem*

(1.1)–(1.2) *satisfying*

$$\varepsilon \doteq \|q_0 - q^{(sol),M}(x,0)\|_{H^{4,4}(\mathbb{R})} < \varepsilon_0, \tag{1.8}$$

q_0 generates scattering data $\{r', \{z_j', c_j'\}_{j=0}^{N-1}\}$ with $N \geq M$. For the two discrete spectrum we use the same order as in (1.4). Suppose M poles in the discrete spectrum of q_0 are close to that of $q^{(sol),M}(x,t)$. The remaining are close to ± 1. It is to say, there exists an index $L \in \{0,\dots,N-1\}$ with $L+M \leq N-1$, so that we have

$$\max_{0\leq j\leq M-1}(|z_j - z_{j+L}'| + |c_j - c_{j+L}'|) + \max_{j>M+L}|1+z_j'| + \max_{j<L}|1-z_j'| < C\varepsilon. \tag{1.9}$$

Define $\xi = x/t$ and let $\xi_0 \in (0,2)$ so that $\{\operatorname{Re} z_j\}_{j=0}^{M-1} \subset [0, \frac{\xi_0}{2})$. Then we have constants $t_0(q_0,\xi_0) > 0$, $C = C(q_0,\xi_0) > 0$ and $\{x_{k+L}\}_{k=0}^{M-1} \subset \mathbb{R}$ such that for $t > t_0(q_0,\xi_0)$, $|\xi+4| < \xi_0$,

$$\left| q(x,t) - \left[-1 + \sum_{j=0}^{M-1} [sol(z_{j+L}'; x - x_{j+L}, t) + 1] \right] \right| \leq Ct^{-1}. \tag{1.10}$$

2. Preliminaries

In this section we provide an review of the results on the direct and inverse scattering problem for the mKdV equation (1.1).

In Section 2.1, we review the Lax pair of formulation of the mKdV and equation, and we present the properties of the Jost solutions. In Section 2.2, we introduce the basic RH problem, which serves as the basis of the nonlinear steepest approach. In Section 2.3, we plot the signature tables of the saddle functions so that we can open $\bar\partial$ lens in accordance with the corresponding decay regions as t sufficiently large.

2.1. Lax pair and Jost solutions

The defocusing mKdV equation (1.1) is the compatibility condition of the following Lax pair

$$\psi_x = X\psi, \quad \psi_t = T\psi, \tag{2.1}$$

where $\psi = \psi(\lambda;x,t)$ is a matrix-valued eigenfunction, $\lambda \in \mathbb{C}$ is the spectral parameter, and

$$X = X(\lambda;x,t) = \mathrm{i}\lambda\sigma_3 + Q, \qquad T = T(\lambda;x,t) = 4\lambda^2 X - 2\mathrm{i}\lambda\sigma_3(Q_x - Q^2) + 2Q^3 - Q_{xx}, \tag{2.2}$$

with $Q = \begin{pmatrix} 0 & q(x,t) \\ q(x,t) & 0 \end{pmatrix}$.

Existence and differentiability of Jost functions. Taking the non-zero boundary conditions (1.2), we then get the asymptotic spectral problems

$$\phi_x^\pm = X_\pm \phi^\pm, \quad \phi_t^\pm = T_\pm \phi^\pm, \tag{2.3}$$

with $X_\pm(z;x) = \mathrm{i}\lambda\sigma_3 + Q_\pm$, $T_\pm(z;x) = (4\lambda^2+2)X_\pm$, and $Q_\pm = \pm\sigma_1$.

The asymptotic eigenvector matrix is given by

$$Y_\pm(z) = I \mp \frac{1}{z}\sigma_2, \tag{2.4}$$

where I denotes the 2×2 identity matrix, and z is the uniformization variable defined as $z = \lambda+\zeta$, with $\lambda(z) = \frac{1}{2}(z+z^{-1})$, and $\zeta(z) = \frac{1}{2}(z-z^{-1})$. For reference, note that $\det Y_\pm(z) = 1 - \frac{1}{z^2}$.

As usual, we define the Jost eigenfunctions $\psi^\pm(z;x)$ as the solutions of the scattering problem such that

$$\psi^\pm(z;x) = Y_\pm(z) \qquad e^{\mathrm{i}\zeta(z)x\sigma_3} + o(1), \qquad x \to \pm\infty.$$

Subsequently, the modified eigenfunctions $\mu^\pm(z;x)$ can be given by factorizing the asymptotic exponential oscillations:

$$\mu^\pm(z;x) = \psi^\pm(z;x)e^{-\mathrm{i}\zeta(z)x\sigma_3}. \tag{2.5}$$

Furthermore, $\mu^\pm(z;x)$ can be defined by the following Volterra integral equations

$$\mu^\pm(z;x) = \begin{cases} Y_\pm(z) + \int_{\pm\infty}^x Y_\pm(z)e^{\mathrm{i}\zeta(z)(x-y)\hat\sigma_3}\left[Y_\pm^{-1}(z)\Delta Q_\pm(y)\mu_\pm(z;y)\right]dy, & z \neq \pm 1, \\ Y_\pm(z) + \int_{\pm\infty}^x \left[I + (x-y)(Q_\pm \pm \mathrm{i}\sigma_3)\right]\Delta Q_\pm(y)\mu_\pm(z;y)dy, & z = \pm 1, \end{cases}$$

where $\Delta Q_\pm = Q - Q_\pm$. Hereafter, we use $\mu_i^\pm(z;x)$ to denote the ith column of $\mu^\pm(z;x)$. Below we just take a quick review on some existing properties for the Jost functions $\mu^\pm(z;x)$, which can be shown in similar way to Ref. [27].

Proposition 2.1. *Given $n \in \mathbb{N}_0$, let $q(x) \mp 1 \in L^{1,n+1}(\mathbb{R}^\pm)$, $q'(x) \in W^{1,1}(\mathbb{R})$.*

- *For $z \in \mathbb{C}\setminus\{0\}$, $\mu_1^+(z;x)$ and $\mu_2^-(z;x)$ can be analytically extended to $\mathbb{C}^+$ and continuously extended to $\mathbb{C}^+ \cup \mathbb{R}$; $\mu_1^-(z;x)$ and $\mu_2^+(z;x)$ can be analytically extended to $\mathbb{C}^-$ and continuously extended to $\mathbb{C}^- \cup \mathbb{R}$.*
- *The maps $q(x) \to \frac{\partial^n}{\partial z^n}\mu_i^\pm(z)$ $(i = 1,2)$ are Lipschitz continuous, specifically, for any $x_0 \in \mathbb{R}$, $\mu_1^-(z)$ and $\mu_2^+(z)$ are continuously differentiable mappings:*

$$\partial_z^n \mu_1^- : \bar{\mathbb{C}}^- \setminus \{0\} \to L_{loc}^\infty\{\bar{\mathbb{C}}^- \setminus \{0\}, C^1((-\infty,x_0],\mathbb{C}^2) \cap W^{1,\infty}((-\infty,x_0],\mathbb{C}^2)\}, \tag{2.6}$$

$$\partial_z^n \mu_2^+ : \bar{\mathbb{C}}^- \setminus \{0\} \to L_{loc}^\infty\{\bar{\mathbb{C}}^- \setminus \{0\}, C^1([x_0,\infty),\mathbb{C}^2) \cap W^{1,\infty}([x_0,\infty),\mathbb{C}^2)\}, \tag{2.7}$$

$\mu_1^+(z)$ and $\mu_2^-(z)$ are continuously differentiable mappings:

$$\partial_z^n \mu_1^+ : \bar{\mathbb{C}}^+ \setminus \{0\} \to L_{loc}^\infty\{\bar{\mathbb{C}}^+ \setminus \{0\}, C^1([x_0,\infty),\mathbb{C}^2) \cap W^{1,\infty}([x_0,\infty),\mathbb{C}^2)\}, \tag{2.8}$$

$$\partial_z^n \mu_2^+ : \bar{\mathbb{C}}^+ \setminus \{0\} \to L_{loc}^\infty\{\bar{\mathbb{C}}^+ \setminus \{0\}, C^1((-\infty,x_0],\mathbb{C}^2) \cap W^{1,\infty}((-\infty,x_0],\mathbb{C}^2)\}. \tag{2.9}$$

- *For the map $q(x) \to \partial_z^n\mu_1^+(z)$, there exists an increasing function $F_n(t)$, such that*

  $$|\partial_z^n\mu_1^+(z)| \leq F_n[(1+|x|)^{n+1}\|q-1\|_{L^{1,n+1}(x,\infty)}], \qquad z \in \bar{\mathbb{C}}^+ \setminus \{0\}.$$

  Additionally, given potentials $q(x)$ and $\widetilde{q}(x)$ close enough, we have

  $$|\partial_z^n(\mu_1^+(z) - \widetilde{\mu}_1^+(z))| \leq \|q-\widetilde{q}\|_{L^{1,n+1}(x,\infty)}F_n[(1+|x|)^{n+1}\|q-1\|_{L^{1,n+1}(x,\infty)}]. \tag{2.10}$$

- *Let S be a compact neighborhood of $\{-1,1\}$ in $\bar{\mathbb{C}}^+ \setminus \{0\}$. Set $x^\pm = \max\{\pm x, 0\}$, then there would be a constant C so that*

  $$|\mu_1^+(z) - (1,z^{-1})^T| \leq C\langle x^-\rangle e^{C\int_x^\infty \langle y-x\rangle|q-1|dy}\|q-1\|_{L^{1,1}(x,\infty)}, \qquad z \in S. \tag{2.11}$$

Asymptotics and symmetries of the Jost functions. The following proposition gives the asymptotic behavior of the Jost functions.

Proposition 2.2. *Suppose that $q(x)\mp 1 \in L^{1,n+1}(\mathbb{R}^\pm)$ and $q'(x) \in W^{1,1}(\mathbb{R})$. Then as $z \to \infty$, we have the asymptotics on $\mathbb{C}^+$*

$$\mu_1^+(z) = e_1 + \frac{1}{z}\begin{pmatrix} -i\int_x^\infty (q^2-1)dx \\ -iq \end{pmatrix} + \mathcal{O}(z^{-2}),$$

$$\mu_2^-(z) = e_2 + \frac{1}{z}\begin{pmatrix} iq \\ i\int_{-\infty}^x (q^2-1)dx \end{pmatrix} + \mathcal{O}(z^{-2}),$$

and on $\mathbb{C}^-$

$$\mu_1^-(z) = e_1 + \frac{1}{z}\begin{pmatrix} -i\int_{-\infty}^x (q^2-1)dx \\ -iq \end{pmatrix} + \mathcal{O}(z^{-2}),$$

$$\mu_2^+(z) = e_2 + \frac{1}{z}\begin{pmatrix} iq \\ i\int_x^\infty (q^2-1)dx \end{pmatrix} + \mathcal{O}(z^{-2}).$$

For $z \in \mathbb{C}^+$, we have the following asymptotics as $z \to 0$,

$$\mu_1^+(z) = -\frac{i}{z}e_2 + \mathcal{O}(1), \quad \mu_2^-(z) = -\frac{i}{z}e_1 + \mathcal{O}(1); \tag{2.12}$$

and for $z \in \mathbb{C}^-$,

$$\mu_1^-(z) = \frac{i}{z}e_2 + \mathcal{O}(1), \quad \mu_2^+(z) = \frac{i}{z}e_1 + \mathcal{O}(1). \tag{2.13}$$

Abel's theorem indicates that for any solution $\psi(z,x)$ of (2.1), one has $\partial_x(\det\psi) = \partial_t(\det\psi) = 0$. Thus, both ψ_- and ψ_+ are two fundamental matrix solutions of the scattering problem with $\det\psi^\pm(z) = 1 - z^{-2}$, and satisfy the linear relation:

$$\psi^+(z;x) = \psi^-(z;x)S(z), \quad z \in \mathbb{R}\setminus\{\pm1,0\}, \tag{2.14}$$

where $S(z)$ is called scattering matrix and is represented as:

$$S(z) = \begin{pmatrix} a(z) & c(z) \\ b(z) & d(z) \end{pmatrix}.$$

Hereafter, we use a bar to denote the complex conjugate. Then we can establish the following symmetries for the Jost functions and the scattering matrix, which will enable us to set up an RH problem with desirable symmetries.

Proposition 2.3. *Suppose that $q(x) \mp 1 \in L^{1,n+1}(\mathbb{R}^\pm)$ and $q'(x) \in W^{1,1}(\mathbb{R})$, then*

1. *For $z \in \bar{\mathbb{C}}^+ \setminus \{0\}$, the Jost functions $\psi_j^\pm$ $(j = 1,2)$ satisfy the symmetries*

$$\psi_1^\pm(z) = \sigma_1\overline{\psi_2^\pm(\bar z)}, \quad \psi_2^\pm(z) = \sigma_1\overline{\psi_1^\pm(\bar z)}. \tag{2.15}$$

$$\psi_1^\pm(z) = \overline{\psi_1^\pm(-\bar z)}, \quad \psi_2^\pm(z) = \overline{\psi_2^\pm(-\bar z)}. \tag{2.16}$$

$$\psi_1^\pm(z) = \mp\frac{i}{z}\psi_2^\pm(\frac{1}{z}), \quad \psi_2^\pm(z) = \pm\frac{i}{z}\psi_1^\pm(\frac{1}{z}). \tag{2.17}$$

2. *The scattering data $a(z), b(z), c(z)$ and $d(z)$ satisfy the symmetries*

$$S(z) = \sigma_1\overline{S(\bar z)}\sigma_1 = -\sigma_2 S(z^{-1})\sigma_2, \quad S(z) = \overline{S(-\bar z)}. \tag{2.18}$$

It then follows that the scattering matrix $S(z)$ can be rewritten as

$$S(z) = \begin{pmatrix} a(z) & \overline{b(z)} \\ b(z) & \overline{a(z)} \end{pmatrix}, \quad z \in \mathbb{R}\setminus\{\pm1,0\}.$$

Scattering map from initial data to reflection coefficient. The reflection coefficient that will be used in the inverse problem is defined by the scattering coefficients $a(z)$ and $b(z)$

$$r(z) \doteq \frac{b(z)}{a(z)}. \tag{2.19}$$

The following proposition provides some useful properties of $a(z)$ and $b(z)$.

Proposition 2.4. *Let $q(x) \mp 1 \in L^{1,n+1}(\mathbb{R}^\pm)$ and $q'(x) \in W^{1,1}(\mathbb{R})$, then*

1. *The scattering coefficients can be expressed as*

$$a(z) = \frac{\mathrm{Wr}(\psi_1^+,\psi_2^-)}{1-z^{-2}}, \quad b(z) = \frac{\mathrm{Wr}(\psi_1^-,\psi_1^+)}{1-z^{-2}}. \tag{2.20}$$

Thus it follows from the analyticities of $\psi^\pm$ that $a(z)$ is analytic in $\mathbb{C}^+$ while $b(z)$ and $r(z)$ are defined on $\mathbb{R}\setminus\{\pm1,0\}$.

2. *For $z \in \mathbb{R}\setminus\{\pm1,0\}$, we have*

$$|a(z)|^2 - |b(z)|^2 = 1, \tag{2.21}$$

which gives a constraint

$$|r(z)|^2 = 1 - \frac{1}{|a(z)|^2} < 1. \tag{2.22}$$

3. *The scattering data has the following asymptotics*

$$\lim_{z\to\infty}(a(z)-1)z = i\int_{\mathbb{R}}(q^2-1)dx, \quad z \in \bar{\mathbb{C}}^+, \tag{2.23}$$

$$\lim_{z\to0}(a(z)+1)z^{-1} = i\int_{\mathbb{R}}(q^2-1)dx, \quad z \in \bar{\mathbb{C}}^+ \tag{2.24}$$

$$|b(z)| = \mathcal{O}(|z|^{-2}), \quad \textit{as } |z| \to \infty, \quad z \in \mathbb{R} \tag{2.25}$$

$$|b(z)| = \mathcal{O}(|z|^2), \quad \textit{as } |z| \to 0, \quad z \in \mathbb{R}. \tag{2.26}$$

So that

$$r(z) \sim z^{-2}, \quad |z| \to \infty; \quad r(z) \sim z^2, \quad |z| \to 0. \tag{2.27}$$

It can also be shown that $z = \pm1$ are the simple poles of $a(z)$ and $b(z)$ on account of the expressions in (2.20). Even though, we still can derive the boundness of $r(z)$ at $z = \pm1$. Using the symmetry in (2.17), it is easy to verify that $\psi_1^-(\pm1) = \pm\mathrm{i}\psi_2^-(\pm1)$, which implies that

$$a(z) = \frac{\pm a_\pm}{z\mp1} + \mathcal{O}(1), \quad b(z) = -\frac{\mathrm{i}a_\pm}{z\mp1} + \mathcal{O}(1), \tag{2.28}$$

where

$$a_\pm = \frac{1}{2}\det[\psi_1^+(\pm1),\psi_2^-(\pm1)]. \tag{2.29}$$

Consequently, it is obvious that the reflection coefficient $r(z)$ is bounded at $z = \pm1$ and

$$\lim_{z\to\pm1} r(z) = \mp\mathrm{i}. \tag{2.30}$$

Although the scattering coefficients have some simple poles, given specific conditions on the initial potential, the reflection coefficient will exhibit smoothness and decay.

Proposition 2.5. *Suppose that $q(x) \mp 1 \in L^{1,2}(\mathbb{R}^\pm)$, $q'(x) \in W^{1,1}(\mathbb{R})$, then $r(z) \in H^1(\mathbb{R})$.*

Proof. Proposition 2.4 implies that $a(z)$ and $b(z)$ are continuous when $z \in \mathbb{R}\setminus\{\pm1,0\}$. Then $r(z)$ is continuous for $z \in \mathbb{R}\setminus\{\pm1,0\}$. From (2.27) and (2.30) we know that $r(z)$ is bounded in the small neighborhood of $\{\pm1,0\}$ and $r(z) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Here we just need to prove that $r'(z) \in L^2(\mathbb{R})$. For $\delta_0 > 0$ small, from Proposition 2.1, the maps

$$q \to \det[\psi_1^+(z),\psi_2^-(z)] \quad \text{and} \quad q \to \det[\psi_1^-(z),\psi_1^+(z)] \tag{2.31}$$

are locally Lipschitz maps from

$$\{q : q'(x) \in W^{1,1}(\mathbb{R}) \text{ and } q \in L^{1,n+1}(\mathbb{R})\} \to W^{n,\infty}(\mathbb{R}\setminus(-\delta_0,\delta_0)) \text{ for } n \geq 0. \tag{2.32}$$

According to Proposition 2.1, $q \to \psi_1^+(z,0)$ is a locally Lipschitz map, and this fact also holds for $q \to \psi_2^-(z,0)$ and $q \to \psi_1^-(z,0)$. Together with (2.23)–(2.26), we derive that $q \to r(z)$ is a locally Lipschitz map with values in $W^{n,\infty}(I_{\delta_0}) \cap H^n(I_{\delta_0})$, where $I_{\delta_0} \doteq \mathbb{R}\setminus((-\delta_0,\delta_0)\cup(1-\delta_0,1+\delta_0)\cup(-1-\delta_0,-1+\delta_0))$. Then fix $\delta_0 > 0$ small to make sure that the three intervals $\mathrm{dist}(z,\{\pm1\}) \leq \delta_0$ and $|z| \leq \delta_0$ have no intersection. In the complement of the union

$$|\partial_z^j r(z)| \leq C_{\delta_0}\langle z\rangle^{-1} \text{ for } j = 0,1. \tag{2.33}$$

In the following step, we will just prove the boundedness of $r'(z)$ in the small neighborhood of $z = 1$. Let $z \in U(1,\delta_0)$ be a neighborhood of 1, then we have

$$r(z) = \frac{b(z)}{a(z)} = \frac{\det[\psi_1^-,\psi_1^+]}{\det[\psi_1^+,\psi_2^-]} = \frac{\int_1^z \partial_s\det[\psi_1^-(s),\psi_1^+(s)]ds - 2ia_+}{\int_1^z \partial_s\det[\psi_1^+(s),\psi_1^-(s)]ds + 2a_+}, \tag{2.34}$$

where $a_\pm$ is defined in (2.29). If $a_+ \neq 0$ then obviously $r'(z)$ exist and is bounded near 1. If $a_+ = 0$, then $z = 1$ is not a pole of the scattering data $a(z)$ and $b(z)$. Therefore, they are continuous at $z = 1$, implying

$$r(z) = \frac{\int_1^z \partial_s\det[\psi_1^-(s),\psi_1^+(s)]ds}{\int_1^z \partial_s\det[\psi_1^+(s),\psi_1^-(s)]ds}. \tag{2.35}$$

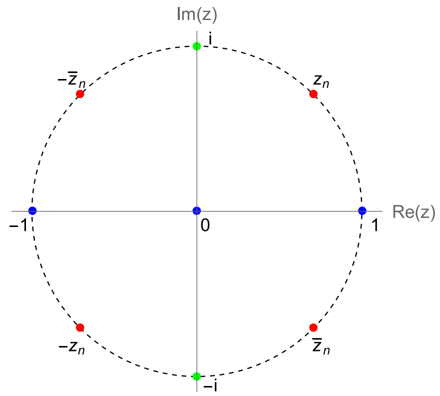


Fig. 2. Distribution of the discrete spectrum on circle $|z|=1$: the jump contour $\mathbb{R}$, the red dots ( ) and green dots ( ) represent the zeros of $a(z)$, they are corresponding to breathers and solitons respectively. The blue dots ( ) represent singularities.

From (2.20) we have that

$$(z^2-1)a(z) = z^2\det[\psi_1^+,\psi_2^-]. \tag{2.36}$$

While $a_+=0$ implies that $\det[\psi_1^+,\psi_2^-]\|_{z=1}=0$, differentiating (2.36) at $z=1$ we get

$$2a(1) = \partial_z\det[\psi_1^+,\psi_2^-]\|_{z=1}. \tag{2.37}$$

With $\|a(1)\|^2 = 1+\|b(1)\|^2 \geq 1$, we have $\partial_z\det[\psi_1^+,\psi_2^-]\|_{z=1}\neq 0$. It then follows that the derivative $r'(z)$ is bounded in a neighborhood of 1.

The proof for the case $z=-1$ is just an analogue. For $z=0$ we need to use $r(z^{-1}) = -\bar{r}(z)$ to illustrate that $r(z)\to 0$. Consequently, $r'(z)\in L^2(\mathbb{R})$. □

The following proposition gives a bound for reflection coefficient, and the proof can be found in [27].

Proposition 2.6. *Suppose that $q(x)\mp 1\in H^{2,2}(\mathbb{R}^\pm)$, then the reflection coefficient satisfies*

$$\|\log(1-|r|^2)\|_{L^p(\mathbb{R})} < \infty \text{ for any } p\geq 1. \tag{2.38}$$

Now we introduce the discrete spectrum. Suppose that $a(z)$ has finite N simple zeros z_k ($k=0,\dots,N-1$) on $D_1 \doteq \{z\in\mathbb{C}^+ : \operatorname{Im} z>0, \operatorname{Re} z\geq 0\}$, then symmetries (2.18) implies that the discrete spectrum appears in double pairs $\mathcal{Z} = \{z_k,\bar{z}_k,-\bar{z}_k,-z_k\}_{k=0}^{N-1}$. When $z_k=-\bar{z}_k$, the corresponding quaternate zeros $\{z_k,\bar{z}_k,-\bar{z}_k,-z_k\}$ degenerate into a pair $\{-i,i\}$, which corresponds to solitons. We further classify these discrete spectrum as

$$\mathcal{Z}_1 \doteq \{z_k\}_{k=0}^{N-1},\quad \mathcal{Z}_2 \doteq \{-\bar{z}_k\}_{k=0}^{N-1},\quad \mathcal{Z}^+ \doteq \mathcal{Z}_1\cup\mathcal{Z}_2,\quad \mathcal{Z}\doteq \mathcal{Z}^+\cup\overline{\mathcal{Z}^+}, \tag{2.39}$$

where $\overline{\mathcal{Z}^+}$ formed by the complex conjugates of $\mathcal{Z}^+$. The discrete spectrum can be seen in Fig. 2.

Next we will state some properties of the zeros of $a(z)$. From the symmetries of $\psi^\pm(z)$ we know that there exists a constant $\gamma_k\in\mathbb{C}$ such that

$$\psi_1^+(z_k) = \gamma_k\psi_2^-(z_k),\qquad \psi_2^+(z_k) = \bar{\gamma}_k\psi_1^-(z_k),\qquad \psi_1^+(z_k) = \gamma_k\psi_2^-(-\bar{z}_k), \tag{2.40}$$

where γ_k is known as the connection coefficient related to z_k.

Proposition 2.7. *Suppose $q(x)\mp 1\in L^{1,2}(\mathbb{R}^\pm)$, then the discrete eigenvalues are simple, finite and distribute on the circle $|z|=1$.*

Proof. Let $z_k\in D_1$ be a zero of $a(z)$. It is well known that the Dirac operator for defocusing case is self-adjoint, meaning that the spectral parameter $\lambda(z_k)$ is real, which leads to the fact that

$$\operatorname{Im}\lambda(z_k) = \frac{|z_k|^2-1}{2|z_k|^2}\operatorname{Im} z_k = 0. \tag{2.41}$$

Therefore, all the zeros of $a(z)$ are distributed on the real axis and the unit circle. Next we prove the zeros are not real. First we note that $z_k\in\mathbb{R}\setminus\{0,\pm 1\}$ cannot be a zero of $a(z)$, as it would contradict (2.21). We will then show that 0 and ± 1 cannot be zeros either. The functions

$$f_k(\theta) = \left|\partial_\theta^k\det[\psi_1^+(e^{i\theta}),\psi_2^-(e^{i\theta})]\right|,\qquad k=0,1 \tag{2.42}$$

are continuous for $\theta\in[0,\pi/2]$. Assume that $z=1$ is an accumulation point, according to the Bolzano–Weierstrass theorem, there exist sequences $\theta_j^{(k)}$, $k=0,1$, satisfying $\lim_{j\to\infty}\theta_j^{(k)}=0$ and $f_k(\theta_j^{(k)})=0$ for each j. From (2.28), $a(z)=o(1)$ as $z\to 1$. This just contradicts the fact in (2.21). The proof is the same when $z=-1$ is an accumulation point. In brief, all the zeros of $a(z)$ are distributed on the unit circle.

From Proposition 2.3, we have the symmetries of $\psi^\pm$ at z_k:

$$\bar{\psi}_1^+(\bar{z}_k^{-1}) = \bar{\gamma}_k\bar{\psi}_2^-(\bar{z}_k^{-1}),\quad \sigma_1\psi_2^+(z_k^{-1}) = \bar{\gamma}_k\sigma_1\psi_1^-(z_k^{-1}),\quad \psi_1^+(z_k) = \bar{\gamma}_k\psi_2^-(z_k),$$

which imply that $\gamma_k=\bar{\gamma}_k$, i.e., $\gamma_k\in\mathbb{R}$. The condition $q(x)\mp 1\in L^{1,1}(\mathbb{R}^\pm)$ guarantees the existence of $\frac{\partial a}{\partial\lambda}$. Evaluating the differentiation of (2.20) at z_k, we have

$$\left.\frac{\partial a}{\partial\lambda}\right|_{z=z_k} = \left.\frac{\det[\partial_\lambda\psi_1^+,\psi_2^-]+\det[\psi_1^+,\partial_\lambda\psi_2^-]}{1-z^{-2}}\right|_{z=z_k}. \tag{2.43}$$

Differentiating the two functions appear in the numerator of (2.43) with respect to x and using the scattering problem (2.1) we obtain that

$$\begin{aligned}\frac{\partial}{\partial x}\det[\partial_\lambda\psi_1^+,\psi_2^-] &= \det[X_\lambda\psi_1^+,\psi_2^-]+\det[X\partial_\lambda\psi_1^+,\psi_2^-]+\det[\partial_\lambda\psi_1^+,X\psi_2^-]\\ &= i\det[\sigma_3\psi_1^+,\psi_2^-],\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial x}\det[\psi_1^+,\partial_\lambda\psi_2^-] &= \det[\psi_1^+,X_\lambda\psi_2^-]+\det[X\psi_1^+,\partial_\lambda\psi_2^-]+\det[\psi_1^+,X\partial_\lambda\psi_2^-]\\ &= i\det[\psi_1^+,\sigma_3\psi_2^-].\end{aligned}$$

According to (2.40) and the decaying properties for each column at z_k, we can easily get that

$$\begin{aligned}\det[\partial_\lambda\psi_1^+,\psi_2^-] &= i\gamma_k\int_{-\infty}^x \det[\sigma_3\psi_2^-(z_k,s),\psi_2^-(z_k,s)]ds,\\ \det[\psi_1^+,\partial_\lambda\psi_2^-] &= i\gamma_k\int_x^{\infty}\det[\sigma_3\psi_2^-(z_k,s),\psi_2^-(z_k,s)]ds.\end{aligned}$$

The symmetries of $\psi^\pm$ give us $\psi_2^-(z_k) = -iz_k^{-1}\sigma_1\overline{\psi_2^-(z_k)}$, which implies

$$\left.\frac{\partial a}{\partial\lambda}\right|_{z=z_k} = \frac{\gamma_k}{2\zeta(z_k)}\int_{\mathbb{R}}|\psi_2^-(z_k)|^2 ds. \tag{2.44}$$

Since γ_k is real and $\zeta(z_k)$ is imaginary, we conclude that $\left.\frac{\partial a}{\partial\lambda}\right|_{z=z_k}\in i\mathbb{R}$. It follows that the zeros of $a(z)$ are simple. □

Now, we can define the trace formula

$$a(z) = \left(\frac{z-i}{z+i}\right)\prod_{n=0}^{N-1}\frac{(z-z_n)(z+\bar{z}_n)}{(z-\bar{z}_n)(z+z_n)}\exp\left(-\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{\log(1-|r(s)|^2)}{s-z}ds\right). \tag{2.45}$$

Remark 2.8. In the trace formula, we claim that $\pm i$ are always present for this paper. To show this, consider the function $a(z) = \left(\frac{z-i}{z+i}\right)^\delta\prod_{n=0}^{N-1}\frac{(z-z_n)(z+\bar{z}_n)}{(z-\bar{z}_n)(z+z_n)}\exp\left(-\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{\log(1-|r(s)|^2)}{s-z}ds\right)$, where $\delta=1$ indicates that $\pm i$ are zeros of $a(z)$ and $\delta=0$ indicates otherwise. From the asymptotic behavior (2.24), we have that $\lim_{z\to 0}a(z)=-1$, which leads to

$$a(0) = -1 = (-1)^\delta\exp\left(-\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{\log(1-|r(s)|^2)}{s}ds\right).$$

Using the symmetries (2.18), which implies $|r(z)|^2 = |r(-z)|^2$, we have that

$$\int_{-\infty}^{0} \frac{\log(1-|r(s)|^2)}{s}\mathrm{d}s = \int_{+\infty}^{0} \frac{\log(1-|r(s)|^2)}{s}\mathrm{d}s.$$

This simplifies the equation for $a(0)$ to $-1 = (-1)^{\delta}$. Therefore, one has $\delta = 1$, meaning that $\pm\mathrm{i}$ are indeed zeros of $a(z)$. However, $\pm\mathrm{i}$ are not always the zeros of $a(z)$ in the generic case. As shown in [31] that if $q_+/q_- = 1$, then $\pm\mathrm{i}$ are no longer zeros of $a(z)$.

At the end of this section, we demonstrate the time evolution of the scattering data $a(z)$, $b(z)$ and $r(z)$. Define $\psi^{\pm}(z;x,t) \doteq \psi^{\pm}(z;x)H(z;t)$ as the time-dependent eigenfunctions, where $H(z;t)$ is a function to be determined. Substituting $\psi^{\pm}(z;x,t)$ into (2.1) leads to

$$(\partial_t - T)[\psi^{\pm}(z;x)H(z;t)] = 0. \tag{2.46}$$

As $x \to \pm\infty$, (2.46) gives $H(z;t) = \mathrm{e}^{\mathrm{i}\zeta(z)(4\lambda^2+2)t\sigma_3}$. Differentiating (2.14) with respect to t and using (2.46), we derive an ODE that the scattering matrix satisfies

$$\partial_t S = \mathrm{i}\zeta(z)(4\lambda^2(z)+2)[\sigma_3, S], \tag{2.47}$$

which indicates that

$$a(z,t) = a(z,0), \quad b(z,t) = b(z,0)\mathrm{e}^{\mathrm{i}\zeta(4\lambda^2+2)t}, \quad r(z,t) = r(z,0)\mathrm{e}^{\mathrm{i}\zeta(4\lambda^2+2)t};$$
$$z_k(t) = z_k(0), \quad \gamma_k(t) = \gamma_k(0)\mathrm{e}^{i\zeta(4\lambda^2+2)t}.$$

Hereafter, for convenience, we still use $a(z)$, $b(z)$ and $r(z)$ to denote $a(z,t)$, $b(z,t)$ and $r(z,t)$, respectively.

2.2. An RH characterization of the mKdV equation

Now we introduce the sectionally meromorphic function

$$m(z) = m(z;x,t) = \begin{cases} \left(\dfrac{\mu_1^+(z;x,t)}{a(z)}, \mu_2^-(z;x,t)\right), & z \in \mathbb{C}^+, \\ \left(\mu_1^-(z;x,t), \dfrac{\mu_2^+(z;x,t)}{\overline{a(\bar z)}}\right), & z \in \mathbb{C}^-, \end{cases} \tag{2.48}$$

then the following matrix RH problem is proposed.

RH Problem 2.1. *Find a 2×2 matrix-valued function $m(z;x,t)$ such that*

1. *$m(z)$ is meromorphic for $z \in \mathbb{C}\setminus\mathbb{R}$, with poles belonging to the set $\mathcal{Z}$ defined in (2.39).*
2. *$m(z)$ satisfies the following symmetries*
$$m(z) = \sigma_1\overline{m(\bar z)}\sigma_1 = z^{-1}m(z^{-1})\sigma_2 = \overline{m(-\bar z)}. \tag{2.49}$$
3. *$m(z)$ has the following asymptotics*
$$m(z;x,t) = I + \mathcal{O}(z^{-1}), \quad z\to\infty; \qquad zm(z;x,t) = \sigma_2 + \mathcal{O}(z), \quad z \to 0.$$
4. *$m_{\pm}(z;x,t) = \lim_{\mathbb{C}^{\pm}\ni z'\to z} m(z';x,t)$ exist for any $z \in \mathbb{R}\setminus\{0\}$ and meet the jump relation $m_+(z;x,t) = m_-(z;x,t)V(z)$, where*
$$V(z) = \begin{pmatrix} 1-|r(z)|^2 & -\overline{r(z)}\mathrm{e}^{2it\theta(z)} \\ r(z)\mathrm{e}^{-2it\theta(z)} & 1\end{pmatrix}, \tag{2.50}$$
and
$$\theta(z) = \zeta(z)\{x/t + 4\lambda^2(z) + 2\}. \tag{2.51}$$
5. *$m(z;x,t)$ has residue conditions at the simple poles in $\mathcal{Z} = \mathcal{Z}^+ \cup \overline{\mathcal{Z}^+}$*
$$\begin{aligned}
&\mathrm{Res}_{z=z_k}\, m(z;x,t) = \lim_{z\to z_k} m(z;x,t)\begin{pmatrix} 0 & 0 \\ c_k(x,t) & 0\end{pmatrix}, \\
&\mathrm{Res}_{z=\bar z_k}\, m(z;x,t) = \lim_{z\to \bar z_k} m(z;x,t)\begin{pmatrix} 0 & \bar c_k(x,t) \\ 0 & 0\end{pmatrix}, \\
&\mathrm{Res}_{z=-\bar z_k}\, m(z;x,t) = \lim_{z\to -\bar z_k} m(z;x,t)\begin{pmatrix} 0 & 0 \\ -\bar c_k(x,t) & 0\end{pmatrix}, \\
&\mathrm{Res}_{z=-z_k}\, m(z;x,t) = \lim_{z\to -z_k} m(z;x,t)\begin{pmatrix} 0 & -c_k(x,t) \\ 0 & 0\end{pmatrix}.
\end{aligned} \tag{2.52}$$
where
$$\begin{aligned} c_k(x,t) &= \frac{\gamma_k(0)}{a'(z_k)}\mathrm{e}^{-2it\theta(z_k)} \doteq c_k\mathrm{e}^{-2it\theta(z_k)}, \\ c_k &= \frac{\gamma_k(0)}{a'(z_k)} = \frac{2z_k}{\int_{\mathbb{R}}|\psi_2^+(z_k;x,0)|^2dx} = z_k|c_k|. \end{aligned} \tag{2.53}$$

Remark 2.9. The solution of RH Problem 2.1 preserves the symmetry from z to $-z$, ensuring that the potential recovered from this solution is real, consistent with the fact that the solution of the mKdV equation is real.

It then follows the reconstruction formula

$$q(x,t) = \lim_{z\to\infty} \mathrm{i}zm_{21}(z;x,t). \tag{2.54}$$

2.3. Signature tables

Large-time asymptotic behavior of RH Problem 2.1 is influenced by decay/growth of oscillatory terms $\mathrm{e}^{\pm 2\mathrm{i}t\theta(z)}$ and phase points of $\theta(z)$. Let $\xi = x/t$, direct calculation gives

$$\theta'(z) = \frac{3}{2}z^2 + \frac{\xi+3}{2z^2} + \frac{3}{2z^4} + \frac{\xi+3}{2},$$

from which we can get six phase points. Moreover, he decay/growth of oscillatory terms $\mathrm{e}^{\pm 2\mathrm{i}t\theta(z)}$ is determined by the sign of $\mathrm{Re}(2\mathrm{i}t\theta(z))$. The decaying region of $\mathrm{Re}(2\mathrm{i}t\theta(z))$ is shown in Fig. 4.

Proposition 2.10 (*Distribution of Saddle Points*)**.** *In addition to the two fixed saddle points $\pm i$, there exist four saddle points satisfying the following properties for different ξ (see Fig. 3): - For $\xi < -6$, the four saddle points $\xi_j := \xi_j(\xi)$, $j = 1,2,3,4$ lie on the jump contour $\Sigma = \mathbb{R}\setminus\{0\}$. Moreover, we have $\xi_4 < -1 < \xi_3 < 0 < \xi_2 < 1 < \xi_1$ and $\xi_1 = \frac{1}{\xi_2} = -\frac{1}{\xi_3} = -\xi_4$; - For $-6 < \xi < 6$, the four saddle points are away from the coordinate axes; - For $\xi > 6$, the four saddle points lie on the imaginary axis with $\mathrm{Im}\,\xi_1 > 1 > \mathrm{Im}\,\xi_2 > 0 > \mathrm{Im}\,\xi_3 > -1 > \mathrm{Im}\,\xi_4$ and $\xi_1\xi_2 = \xi_3\xi_4 = -1$.*

Proof. Letting $\theta'(z) = 0$ and factoring the left-hand side gives us

$$(1+z^2)\left(3z^4 + \xi z^2 + 3\right) = 0. \tag{2.55}$$

Solving for z, we have $z = \pm\mathrm{i}$ and

$$z^2 = -\frac{\xi + \sqrt{\xi^2-36}}{6}, \quad \text{or} \quad z^2 = -\frac{\xi - \sqrt{\xi^2-36}}{6}. \tag{2.56}$$

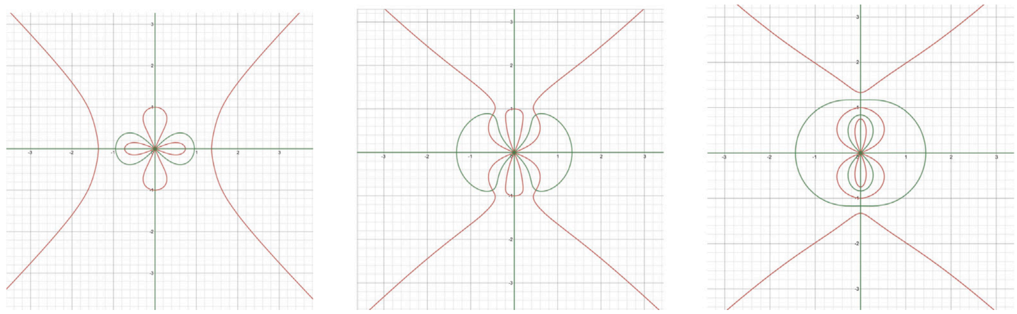
For $\xi < -6$, we have $-\frac{\xi\pm\sqrt{\xi^2-36}}{6} > 0$, and the four roots are as follows.

$$\xi_1 = \sqrt{-\frac{\xi - \sqrt{\xi^2-36}}{6}}, \quad \xi_4 = -\sqrt{-\frac{\xi - \sqrt{\xi^2-36}}{6}}, \tag{2.57}$$

$$\xi_2 = \sqrt{-\frac{\xi + \sqrt{\xi^2-36}}{6}}, \quad \xi_3 = -\sqrt{-\frac{\xi + \sqrt{\xi^2-36}}{6}}. \tag{2.58}$$

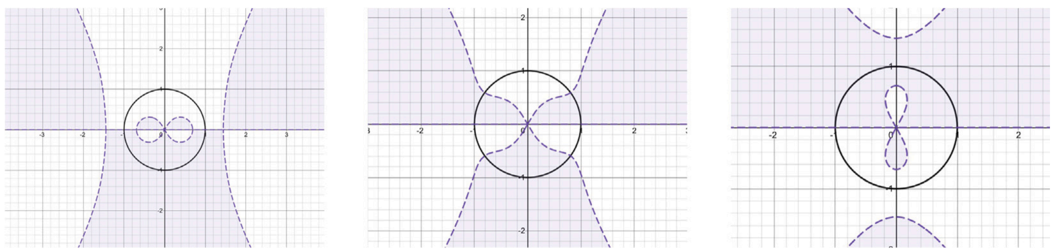
with $\xi_4 < -1 < \xi_3 < 0 < \xi_2 < 1 < \xi_1$ and $\xi_1 = \frac{1}{\xi_2} = -\frac{1}{\xi_3} = -\xi_4$.

For $-6 < \xi < 6$, the discriminant $\xi^2 - 36$ is less than zero. Therefore, there exist four saddle points $\xi_j = \mathrm{Re}(\xi_j) + \mathrm{i}\,\mathrm{Im}(\xi_j)$, where $\mathrm{Re}(\xi_j), \mathrm{Im}(\xi_j) \neq 0$, $j = 1,2,3,4$.



(a) (b) (c)

Fig. 3. Plots of the distributions for saddle points: (**a**) $\xi < -6$, (**b**) $-6 < \xi < 6$, (**c**) $\xi > 6$. The red curve represents $\mathrm{Re}\,\theta'(z) = 0$, and the green curve represents $\mathrm{Im}\,\theta'(z) = 0$. The intersection points are the saddle points which represent the zeros of $\theta'(z) = 0$.



(a) (b) (c)

Fig. 4. Signature table of $\mathrm{Re}(2\mathrm{i}t\theta(z))$ with different ξ: (**a**) $\xi < -6$, (**b**) $-6 < \xi < -2$, (**c**) $\xi > -2$. In the purple region, $\mathrm{Re}(2\mathrm{i}t\theta) < 0$, while in the white region, $\mathrm{Re}(2\mathrm{i}t\theta) > 0$. The purple dashed curves are the critical curves.

For $\xi > 6$, we have $-\frac{\xi\pm\sqrt{\xi^2-36}}{6} < 0$, and the four pure imaginary saddle points are as follows

$$\xi_1 = \mathrm{i}\sqrt{\frac{\xi+\sqrt{\xi^2-36}}{6}}, \quad \xi_4 = -\mathrm{i}\sqrt{\frac{\xi+\sqrt{\xi^2-36}}{6}}, \tag{2.59}$$

$$\xi_2 = \mathrm{i}\sqrt{\frac{\xi-\sqrt{\xi^2-36}}{6}}, \quad \xi_3 = -\mathrm{i}\sqrt{\frac{\xi-\sqrt{\xi^2-36}}{6}}, \tag{2.60}$$

with $\mathrm{Im}\,\xi_1 > 1 > \mathrm{Im}\,\xi_2 > 0 > \mathrm{Im}\,\xi_3 > -1 > \mathrm{Im}\,\xi_4$ and $\xi_1\xi_2 = \xi_3\xi_4 = -1$. □

According to Figs. 3 and 4, it could be observed the following:

- For $\xi < -6$, there are four stationary phase points in addition to $i, -i$, all of which are located in the jump contour as shown in Fig. 3(a), with the corresponding signature table in Fig. 4(a).
- For $-6 < \xi < -2$, The distribution of phase points is shown in Fig. 3(b), with the signature table shown in Fig. 4(b).
- For $\xi > -2$, there exist again four stationary phase points besides $i, -i$.
  - When $-2 < \xi < 6$, the four saddle points are away from the coordinate axis (both real and imaginary axis), corresponding to Fig. 3(b), with the signature table shown in Fig. 4(c). The asymptotic analysis for $-2 < \xi < 6$ could be seen as a specific case of the analysis for $-6 < \xi < -2$.
  - For $\xi > 6$, the four saddle points are all distributed on the imaginary axis as shown in Fig. 3(c), and the signature table is still shown in Fig. 4(c).

3. Deformation of the RH problem

In this section, two crucial steps are given to use the steepest descent method: (i) converting the residue conditions of those poles, which are far away from the critical lines, into jump conditions on new auxiliary contours; (ii) using a well-known factorization for the jump matrix to deform the jump line $\mathbb{R}$ into such lines that the jump matrices on them decay exponentially as $t \to \infty$.

3.1. Interpolation and conjugation

By simple calculation, we find that on the unit circle the phase function satisfies

$$\mathrm{Re}(2\mathrm{i}t\theta(z)) = -2t\sin\omega[\xi + 2 + 4\cos^2\omega], \tag{3.1}$$

where ω is the argument of $z = \mathrm{e}^{\mathrm{i}\omega}$.

For $-6 < \xi < -2$, let $\xi_0 = \sqrt{-\frac{\xi+2}{4}}$, then the discrete spectrum $\mathcal{Z}_1$ defined in (2.39) decomposes into three sets: points with $\mathrm{Re}(z_k) > \xi_0$ and exponentially decaying connection coefficients $c_k(x,t) = c_k \mathrm{e}^{-2\mathrm{i}t\theta(z_k)}$ as $t \to \infty$; points with $0 \le \mathrm{Re}(z_k) < \xi_0$ and growing connection coefficients; and a single point with $\mathrm{Re}(z_k) = \xi_0$ and a bounded connection coefficient in t (see Fig. 4(b)). Let $\rho > 0$ sufficiently small such that

$$\rho < \frac{1}{2}\min\left\{\min_{z_j, z_k \in \mathcal{Z}^+} |\mathrm{Re}(z_j - z_k)|, \min_{z_k\in\mathcal{Z}^+} |\mathrm{Im}(z_k)|\right\}.$$

We divide the index set $H \doteq \{0, 1, \ldots, N-1\}$ into two subsets

$$\vartriangle = \left\{j \in H : \mathrm{Re}(z_j) > \xi_0\right\}, \qquad \nabla = \left\{j \in H : 0 \le \mathrm{Re}(z_j) \le \xi_0\right\},$$

and define

$$\Lambda = \left\{j \in H : |\mathrm{Re}(z_j) - \xi_0| < \rho \text{ and } |\mathrm{Re}(\bar{z}_j) + \xi_0| < \rho\right\}, \tag{3.2}$$

then the sets $|\mathrm{Re}(z - z_j)| \le \rho$ are pairwise disjoint. If $j_0 \in \Lambda \neq \emptyset$, then we have $|\mathrm{e}^{\mathrm{i}t\theta(z_{j_0})}| = \mathcal{O}(1)$.

Note that we can use a well-known factorization to deform the jump matrix V:

$$V(z) = \begin{pmatrix} 1 - |r(z)|^2 & -\overline{r(z)}\mathrm{e}^{2\mathrm{i}t\theta(z)} \\ r(z)\mathrm{e}^{-2\mathrm{i}t\theta(z)} & 1 \end{pmatrix} = B(z)T_0(z)B(z)^{-\dagger}, \tag{3.3}$$

where

$$B(z)=\begin{pmatrix}1 & 0\\ \frac{r(z)}{1-|r(z)|^2}e^{-2it\theta} & 1\end{pmatrix},\quad T_0(z)=(1-|r|^2)^{\sigma_3},\quad B(z)^{-\dagger}=\begin{pmatrix}1 & \frac{-\bar r(z)}{1-|r(z)|^2}e^{2it\theta}\\ 0 & 1\end{pmatrix},$$

and $B^{\dagger}$ is the Hermitian conjugate of B. Therefore, one can extend $B(z)$ into $\mathbb{C}^-$ and $B^{-\dagger}$ into $\mathbb{C}^+$, while the second term T_0 remains on $\mathbb{R}$. This deformation is helpful when the factors into regions in which the corresponding off-diagonal exponential terms $e^{\pm 2it\theta}$ are decaying. We then start to handle the second term T_0.

Define the function

$$T(z;\xi)=-\prod_{k\in\triangle}\frac{(z-z_k)(z+\bar z_k)}{(zz_k-1)(z\bar z_k+1)}\exp\left(-\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\log(1-|r(s)|^2)\left(\frac{1}{s-z}-\frac{1}{2s}\right)ds\right). \tag{3.4}$$

Proposition 3.1. *$T(z;\xi)$ is a meromorphic function in $\mathbb{C}\setminus\mathbb{R}$ and has the following properties:*

- *$T(z)$ has simple poles at $\pm z_k$ and simple zeros at $\pm\bar z_k$ with $\mathrm{Re}(z_k)>\xi_0$.*
- *$T(z)$ satisfies the jump condition*
$$T_-(z;\xi)=T_+(z;\xi)(1-|r(z)|^2),\qquad z\in\mathbb{R}. \tag{3.5}$$
- *$T(z)$ has the following symmetries*
$$\overline{T(\bar z;\xi)}=T^{-1}(z;\xi)=T(z^{-1};\xi)=T(-z;\xi). \tag{3.6}$$
- *$T(z)$ has the following asymptotics:*
$$T(\infty;\xi)=\lim_{z\to\infty}T(z;\xi)=(-1)^{|\triangle|},\qquad z\to\infty, \tag{3.7}$$
where $|\triangle|$ is the cardinality of the set $\triangle$, and $|T(\infty,\xi)|=1$.
- *As $z\to\infty$, we have the asymptotic expansion:*
$$T(z;\xi)=T(\infty;\xi)\left(I-\frac{1}{z}\left(\sum_{k\in\triangle}4i\,\mathrm{Im}(z_k)-\frac{1}{2\pi i}\int_{\mathbb{R}}\log(1-|r(s)|^2)ds\right)+o(z^{-1})\right). \tag{3.8}$$
- *$\frac{a(z)}{T(z;\xi)}$ is holomorphic in $\mathbb{C}^+$ and there exists a constant $C(q_0)$ so that*
$$\left|\frac{a(z)}{T(z;\xi)}\right|<C(q_0),\qquad z\in\mathbb{C}^+. \tag{3.9}$$
Moreover, $\frac{a(z)}{T(z;\xi)}$ can be continuously extended to $\mathbb{R}$, with $|\frac{a(z)}{T(z;\xi)}|=1$ for $z\in\mathbb{R}$.

Proof. The first three properties are easy to prove according to the definition of $T(z)$, we begin from the fourth. As $z\to\infty$, the product

$$\prod_{k\in\triangle}\frac{(z-z_k)(z+\bar z_k)}{(zz_k-1)(z\bar z_k+1)}=\prod_{k\in\triangle}\frac{1}{z_k}\left(\frac{z-z_k}{z-\bar z_k}\right)\frac{-1}{\bar z_k}\left(\frac{z+\bar z_k}{z+z_k}\right)\to(-1)^{|\triangle|},$$

together with $\frac{1}{s-z}-\frac{1}{2s}\to-\frac{1}{2s}$ and $\int_{\mathbb{R}}\frac{\log(1-|r(s)|^2)}{2s}ds=0$ implies the fourth property. The next property is a simple corollary of the last one. In the end, from the trace formula (2.45) we have

$$\frac{a(z)}{T(z)}=-\prod_{k\in\nabla}\frac{(z-z_k)(z+\bar z_k)}{(z-\bar z_k)(z+z_k)}. \tag{3.10}$$

Note that the absolute value in the right-hand-side is not larger than 1 for $z\in\mathbb{C}^+$. So we have proved the proposition. □

Next, we proceed with interpolations and conjugations. We first introduce the interpolation function $G(z)$:

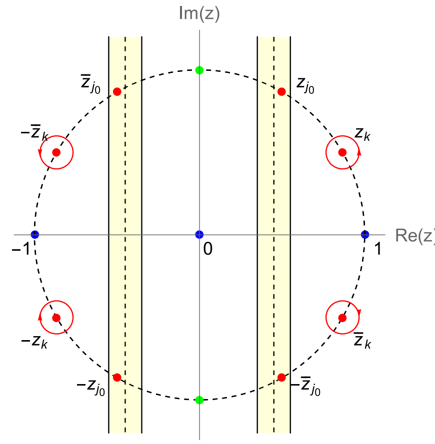


Fig. 5. The dashed lines are where $\mathrm{Re}\,z=\pm\xi_0$. We divide the discrete spectrum in the first quadrant D_1 into three sets: $\triangle\setminus\Lambda$, $\nabla\setminus\Lambda$ and Λ, with the poles reserved in Λ. By the symmetries, the discrete spectrum in the second quadrant is also divided into three sets.

- For $j\in\triangle\setminus\Lambda$,
$$G(z)=\begin{cases}\begin{pmatrix}1 & -\frac{(z-z_j)e^{2it\theta(z_j)}}{c_j}\\ 0 & 1\end{pmatrix}, & |z-z_j|<\rho,\\ \begin{pmatrix}1 & \frac{(z+\bar z_j)e^{2it\theta(z_j)}}{\bar c_j}\\ 0 & 1\end{pmatrix}, & |z+\bar z_j|<\rho,\\ \begin{pmatrix}1 & 0\\ -\frac{(z-\bar z_j)e^{-2it\theta(\bar z_j)}}{\bar c_j} & 1\end{pmatrix}, & |z-\bar z_j|<\rho,\\ \begin{pmatrix}1 & 0\\ \frac{(z+z_j)e^{-2it\theta(\bar z_j)}}{c_j} & 1\end{pmatrix}, & |z+z_j|<\rho;\end{cases} \tag{3.11}$$
- For $j\in\nabla\setminus\Lambda$,
$$G(z)=\begin{cases}\begin{pmatrix}1 & 0\\ -\frac{c_je^{-2it\theta(z_j)}}{z-z_j} & 1\end{pmatrix}, & |z-z_j|<\rho,\\ \begin{pmatrix}1 & 0\\ \frac{\bar c_je^{-2it\theta(z_j)}}{z+\bar z_j} & 1\end{pmatrix}, & |z+\bar z_j|<\rho,\\ \begin{pmatrix}1 & -\frac{\bar c_je^{2it\theta(\bar z_j)}}{z-\bar z_j}\\ 0 & 1\end{pmatrix}, & |z-\bar z_j|<\rho,\\ \begin{pmatrix}1 & \frac{c_je^{2it\theta(\bar z_j)}}{z+z_j}\\ 0 & 1\end{pmatrix}, & |z+z_j|<\rho;\end{cases} \tag{3.12}$$
- Elsewhere, $G(z)=I$.

Then we can introduce the first transformation which converts the poles that are far away from the critical lines into jumps

$$m^{(1)}(z)=T(\infty)^{-\sigma_3}m(z)G(z)T(z)^{\sigma_3}. \tag{3.13}$$

Define the contour

$$\Sigma^{(1)}\doteq\mathbb{R}\cup\left\{\bigcup_{j\in H\setminus\Lambda}\{z\in\mathbb{C}:|z\pm z_j|=\rho\text{ or }|z\pm\bar z_j|=\rho\}\right\}. \tag{3.14}$$

As shown in Fig. 5, the small circles around the poles are oriented counterclockwise in $\mathbb{C}^+$ and clockwise in $\mathbb{C}^-$. Then $m^{(1)}(z)$ satisfies the following RH problem.

RH Problem 3.1. *Find a 2×2 matrix-valued function $m^{(1)}(z;x,t)$ such that*

1. $m^{(1)}(z;x,t)$ *is meromorphic in* $\mathbb{C} \setminus \Sigma^{(1)}$*, where* $\Sigma^{(1)}$ *is defined in* (3.14).
2. $m^{(1)}$ *has the following asymptotics*

$$m^{(1)}(z;x,t) = I + \mathcal{O}(z^{-1}), \ \text{as } z \to \infty,$$
$$zm^{(1)}(z;x,t) = \sigma_2 + \mathcal{O}(z), \ \text{as } z \to 0.$$

3. $m^{(1)}_{\pm}(z;x,t)$ *exist for* $z \in \Sigma^{(1)}$ *and meet the jump relation* $m^{(1)}_{+}(z;x,t) = m^{(1)}_{-}(z;x,t)V^{(1)}(z)$*, where*

   - *for* $z \in \mathbb{R}$,

$$V^{(1)}(z) = \begin{pmatrix} 1 & 0 \\ \frac{r(z)}{1-|r|^2}T_-^{-2}(z)e^{-2it\theta} & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{-\bar{r}(z)}{1-|r|^2}T_+^{-2}(z)e^{2it\theta} \\ 0 & 1 \end{pmatrix}, \tag{3.15}$$

   - *for* $j \in \triangle \setminus \Lambda$,

$$V^{(1)}(z) = \begin{cases} \begin{pmatrix} 1 & -\frac{(z-z_j)e^{2it\theta(z_j)}}{c_j}T^{-2}(z) \\ 0 & 1 \end{pmatrix}, & |z-z_j| = \rho, \\ \begin{pmatrix} 1 & \frac{(z+\bar{z}_j)e^{2it\theta(z_j)}}{\bar{c}_j}T^{-2}(z) \\ 0 & 1 \end{pmatrix}, & |z+\bar{z}_j| = \rho, \\ \begin{pmatrix} 1 & 0 \\ -\frac{(z-\bar{z}_j)e^{-2it\theta(\bar{z}_j)}}{\bar{c}_j}T^{2}(z) & 1 \end{pmatrix}, & |z-\bar{z}_j| = \rho, \\ \begin{pmatrix} 1 & 0 \\ \frac{(z+z_j)e^{-2it\theta(\bar{z}_j)}}{c_j}T^{2}(z) & 1 \end{pmatrix}, & |z+z_j| = \rho, \end{cases} \tag{3.16}$$

   - *for* $j \in \nabla \setminus \Lambda$,

$$V^{(1)}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ -\frac{c_j e^{-2it\theta(z_j)}}{z-z_j}T^{2}(z) & 1 \end{pmatrix}, & |z-z_j| = \rho, \\ \begin{pmatrix} 1 & 0 \\ \frac{\bar{c}_j e^{-2it\theta(z_j)}}{z+\bar{z}_j}T^{2}(z) & 1 \end{pmatrix}, & |z+\bar{z}_j| = \rho, \\ \begin{pmatrix} 1 & -\frac{\bar{c}_j e^{2it\theta(\bar{z}_j)}}{z-\bar{z}_j}T^{-2}(z) \\ 0 & 1 \end{pmatrix}, & |z-\bar{z}_j| = \rho, \\ \begin{pmatrix} 1 & \frac{c_j e^{2it\theta(z_j)}}{z+z_j}T^{-2}(z) \\ 0 & 1 \end{pmatrix}, & |z+z_j| = \rho. \end{cases} \tag{3.17}$$

4. *If there exists a $j_0 \in \Lambda$, then $m^{(1)}(z;x,t)$ satisfies the following residue conditions at $\pm z_{j_0}$ and $\pm \bar{z}_{j_0}$:*
   If $j_0 \in \triangle \cap \Lambda$,

$$\operatorname{Res}_{z=z_{j_0}} m^{(1)}(z) = \lim_{z\to z_{j_0}} m^{(1)}(z)\begin{pmatrix} 0 & c_{j_0}^{-1}e^{2it\theta(z_{j_0})}T'(z_{j_0})^{-2} \\ 0 & 0 \end{pmatrix},$$
$$\operatorname{Res}_{z=\bar{z}_{j_0}} m^{(1)}(z) = \lim_{z\to \bar{z}_{j_0}} m^{(1)}(z)\begin{pmatrix} 0 & 0 \\ \bar{c}_{j_0}^{-1}e^{2it\theta(z_{j_0})}\bar{T}'(z_{j_0})^{-2} & 0 \end{pmatrix},$$
$$\operatorname{Res}_{z=-\bar{z}_{j_0}} m^{(1)}(z) = \lim_{z\to -\bar{z}_{j_0}} m^{(1)}(z)\begin{pmatrix} 0 & -\bar{c}_{j_0}^{-1}e^{2it\theta(z_{j_0})}\bar{T}'(z_{j_0})^{-2} \\ 0 & 0 \end{pmatrix},$$
$$\operatorname{Res}_{z=-z_{j_0}} m^{(1)}(z) = \lim_{z\to -z_{j_0}} m^{(1)}(z)\begin{pmatrix} 0 & 0 \\ -c_{j_0}^{-1}e^{2it\theta(z_{j_0})}T'(z_{j_0})^{-2} & 0 \end{pmatrix}.$$

   If $j_0 \in \nabla \cap \Lambda$,

$$\operatorname{Res}_{z=z_{j_0}} m^{(1)}(z) = \lim_{z\to z_{j_0}} m^{(1)}(z)\begin{pmatrix} 0 & 0 \\ c_{j_0}e^{-2it\theta(z_{j_0})}T(z_{j_0})^{2} & 0 \end{pmatrix},$$
$$\operatorname{Res}_{z=\bar{z}_{j_0}} m^{(1)}(z) = \lim_{z\to \bar{z}_{j_0}} m^{(1)}(z)\begin{pmatrix} 0 & \bar{c}_{j_0}e^{-2it\theta(z_{j_0})}\bar{T}(z_{j_0})^{2} \\ 0 & 0 \end{pmatrix},$$
$$\operatorname{Res}_{z=-\bar{z}_{j_0}} m^{(1)}(z) = \lim_{z\to -\bar{z}_{j_0}} m^{(1)}(z)\begin{pmatrix} 0 & 0 \\ -\bar{c}_{j_0}e^{-2it\theta(z_{j_0})}\bar{T}(z_{j_0})^{2} & 0 \end{pmatrix},$$
$$\operatorname{Res}_{z=-z_{j_0}} m^{(1)}(z) = \lim_{z\to -z_{j_0}} m^{(1)}(z)\begin{pmatrix} 0 & -c_{j_0}e^{-2it\theta(z_{j_0})}T(z_{j_0})^{2} \\ 0 & 0 \end{pmatrix}.$$

5. $m^{(1)}(z)$ *satisfies the symmetries:* $m^{(1)}(z) = \sigma_1\overline{m^{(1)}(\bar{z})}\sigma_1 = z^{-1}m^{(1)}(z^{-1})\sigma_2 = \overline{m^{(1)}(-\bar{z})}$.

Proof. Now we prove that $m^{(1)}(z)$ satisfies the above RH problem. The first statement and the asymptotics as $z \to \infty$ in RH Problem 3.1 can be obtained directly from RH Problem 2.1, we just prove the asymptotics as $z \to 0$. Using the symmetry in (3.6) and the expansion in (3.8), we have

$$\begin{aligned} zm^{(1)} &= T(\infty)^{-\sigma_3}zm(z)T(z)^{\sigma_3} = T(\infty)^{-\sigma_3}(\sigma_2 + \mathcal{O}(z))T(z^{-1})^{-\sigma_3} \\ &= T(\infty)^{-\sigma_3}(\sigma_2 + \mathcal{O}(z))(T(\infty) + \mathcal{O}(z))^{-\sigma_3} = \sigma_2 + \mathcal{O}(z). \end{aligned}$$

The jump relation is also a direct inference of (3.11) and RH Problem 2.1. Now we derive the residue conditions. For $j_0 \in \nabla \cap \Lambda$,

$$\begin{aligned} \operatorname{Res}_{z=z_{j_0}} m^{(1)} &= \operatorname{Res}_{z=z_{j_0}} T(\infty)^{-\sigma_3}m(z)T(z)^{\sigma_3} \\ &= \lim_{z\to z_{j_0}} T(\infty)^{-\sigma_3}m(z)T(z)^{\sigma_3}T(z)^{-\sigma_3}\begin{pmatrix} 0 & 0 \\ c_j e^{-2it\theta(z_{j_0})} & 0 \end{pmatrix}T(z)^{\sigma_3} \\ &= \lim_{z\to z_{j_0}} m^{(1)}(z)\begin{pmatrix} 0 & 0 \\ c_{j_0}e^{-2it\theta(z_{j_0})}T(z_{j_0})^{2} & 0 \end{pmatrix}. \end{aligned}$$

For $j_0 \in \triangle \cap \Lambda$, we have

$$\begin{aligned} \operatorname{Res}_{z=z_{j_0}} m^{(1)} &= \lim_{z\to z_{j_0}} (z-z_{j_0})T(\infty)^{-\sigma_3}\left(\frac{m_1^{+}(z)T(z)}{a(z)}, \frac{m_2^{-}(z)}{T(z)}\right) \\ &= T(\infty)^{-\sigma_3}\left(0, \frac{m_2^{-}(z_{j_0})}{T'(z_{j_0})}\right) \\ &= \lim_{z\to z_{j_0}} T(\infty)^{-\sigma_3}\left(\frac{m_1^{+}(z)T(z)}{a(z)}, \frac{m_2^{-}(z)}{T(z)}\right) \\ &\quad \begin{pmatrix} 0 & c_{j_0}^{-1}e^{2it\theta(z_{j_0})}T'(z_{j_0})^{-2} \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

where we used

$$\begin{aligned} m_1^{(1)}(z_{j_0}) &= T(\infty)^{-\sigma_3}\lim_{z\to z_{j_0}} m_1(z)T(z) = T(\infty)^{-\sigma_3}\lim_{z\to z_{j_0}} [m_1(z)(z-z_{j_0})]\frac{T(z)-T(z_{j_0})}{z-z_{j_0}} \\ &= T(\infty)^{-\sigma_3}\operatorname{Res}_{z=z_{j_0}} m_1(z)T'(z_{j_0}) = T(\infty)^{-\sigma_3}c_{j_0}e^{-2it\theta(z_{j_0})}m_2(z_{j_0})T'(z_{j_0}). \end{aligned}$$

The others can be obtained using a similar method. At last we prove the symmetries for $m^{(1)}(z)$:

$$\overline{m^{(1)}(\bar{z})} = \bar{T}(\infty)^{-\sigma_3}\overline{m}(\bar{z})\overline{T}(\bar{z})^{\sigma_3} = \bar{T}(\infty)^{-\sigma_3}\sigma_1 m(z)\sigma_1 T(z)^{\sigma_3} = \sigma_1 m^{(1)}(z)\sigma_1;$$
$$m^{(1)}(z^{-1}) = T(\infty)^{-\sigma_3}m(z^{-1})T(z^{-1})^{\sigma_3} = zT(\infty)^{-\sigma_3}m(z)\sigma_2 T(z)^{-\sigma_3} = zm^{(1)}(z)\sigma_2,$$
$$\overline{m^{(1)}(-\bar{z})} = \overline{T(\infty)}^{-\sigma_3}\overline{m(-\bar{z})T(-\bar{z})}^{\sigma_3} = T(\infty)^{-\sigma_3}m(z)T(z)^{\sigma_3} = m^{(1)}(z). \qquad \square$$

3.2. Opening $\bar{\partial}$ lenses

In this section, we aim to eliminate the jump on the real axis by choosing a suitable angle, ensuring that the lenses avoid the poles' surrounding disks.

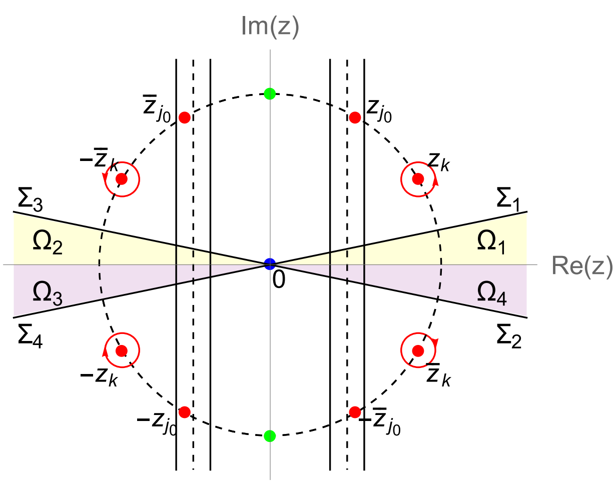


Fig. 6. Find a $\phi(\xi)$ small enough so that there is no pole in the cone and the four rays $\Sigma_k, k = 1,2,3,4$ cannot intersect with any disks $|z \pm z_k| \le \rho$ or $|z \pm \bar{z}_k| \le \rho$.

First we are going to show that there is no phase point on the real axis when $-6 < \xi < -2$.

Proposition 3.2. *When $|\xi + 4| < 2$, there is no phase point in the real axis.*

Proof. From (2.51), we have

$$\theta'(z) = \frac{1}{2}\left\{3(z^2 + \frac{1}{z^4}) + (\xi + 3)(\frac{1}{z^2} + 1)\right\}. \tag{3.18}$$

Assume that $\theta'(z)$ has zeros in the real axis, then

$$3(z^3 + \frac{1}{z^3}) + (\xi + 3)(\frac{1}{z} + z) = 0.$$

Let $s = z + 1/z \in \mathbb{R}$, then $s \in (-\infty, -2] \cup [2, \infty)$ and the above equation becomes $s^3 + (\xi/3 - 2)s = 0$, which means that $s = 0$ or $s^2 = 2 - \xi/3$. While $s^2 \in (4/3, 4)$ as $\xi \in (-6, -2)$, which contradicts the fact that $s^2 \ge 4$. So there is no phase point on the real axis. $\square$

Remark 3.3. The above proof implies that the range for ξ in which there is no phase point on the axis can be extended to $(-6, 6)$. But as $\xi \in (-2, 6)$, Λ will always be an empty set, so here we just investigate the case as $\xi \in (-6, -2)$.

We then define a sufficiently small angle $\theta_0 > 0$ so that the cone $\left\{z \in \mathbb{C} : |\frac{\mathrm{Re}\, z}{z}| > \cos\theta_0\right\}$ has no intersection with the disks $|z \pm z_k| \le \rho$ or $|z \pm \bar{z}_k| \le \rho$. For any $\xi \in (-6, -2)$, let

$$\phi(\xi) = \min\left\{\theta_0, \frac{1}{2}\arccos\frac{-4 - 6\xi - |\xi + 4|}{12}\right\}, \tag{3.19}$$

and $\Omega = \bigcup_{k=1}^{4} \Omega_k$, where

$$\Omega_1 = \{z : \arg z \in (0, \phi(\xi))\}, \qquad \Omega_2 = \{z : \arg z \in (\pi - \phi(\xi), \pi)\},$$
$$\Omega_3 = \{z : \arg z \in (-\pi, -\pi + \phi(\xi))\}, \qquad \Omega_4 = \{z : \arg z \in (-\phi(\xi), 0)\}.$$

Let the boundaries of Ω be

$$\Sigma_1 = e^{i\phi(\xi)}\mathbb{R}^+, \quad \Sigma_2 = e^{i(\pi - \phi(\xi))}\mathbb{R}^+,$$
$$\Sigma_3 = e^{-i(\pi - \phi(\xi))}\mathbb{R}^+, \quad \Sigma_4 = e^{-i\phi(\xi)}\mathbb{R}^+,$$

which can be seen in Fig. 6.

Proposition 3.4. *Let $\xi \doteq \frac{x}{t}$ and $-6 < \xi < -2$. Then for $z = |z|e^{i\omega} = u + iv$ and $F(s) = s + s^{-1}$, the phase function $\theta(z; x, t)$ defined in (2.51) satisfies the following inequalities:*

$$\mathrm{Re}[2it\theta(z; x, t)] \le -\frac{1}{6}F(|z|)^2 t|\sin\omega|(2 - |\xi + 4|), \quad z \in \Omega_1 \cup \Omega_2; \tag{3.20}$$

$$\mathrm{Re}[2it\theta(z; x, t)] \ge \frac{1}{6}F(|z|)^2 t|\sin\omega|(2 - |\xi + 4|), \quad z \in \Omega_3 \cup \Omega_4. \tag{3.21}$$

Proof. We just prove when $z \in \Omega_1$. We can calculate from (2.51), for $z = |z|e^{i\omega}$,

$$\mathrm{Re}(2i\theta) = -F(|z|)\sin\omega[\xi + (F(|z|)^2 - 2)(1 + 2\cos 2\omega)].$$

Note that $F(|z|) \ge 2$, so we have

$$\mathrm{Re}(2i\theta) \le -F(|z|)|\sin\omega|\left[\xi + (F(|z|)^2 - 2)\frac{F(|z|)(2 - |\xi + 4|) - 6\xi}{6(F(|z|)^2 - 2)}\right]$$
$$= -\frac{1}{6}F(|z|)^2|\sin\omega|(2 - |\xi + 4|). \quad \square$$

Proposition 3.5. *Define the functions $R_j : \overline{\Omega}_j \cup \overline{\Omega}_{j+1} \to \mathbb{C}$, $j = 1, 2$, satisfying the following boundary conditions,*

$$\begin{cases} R_1(z) = \dfrac{\bar{r}(z)}{1 - |r(z)|^2}T_+^{-2}(z), & z \in \mathbb{R}; \\ R_1(z) = 0, & z \in \Sigma_1 \cup \Sigma_2; \end{cases}$$

$$\begin{cases} R_2(z) = \dfrac{r(z)}{1 - |r(z)|^2}T_-^{2}(z), & z \in \mathbb{R}; \\ R_2(z) = 0, & z \in \Sigma_3 \cup \Sigma_4. \end{cases}$$

For a fixed constant $c_1(q_0)$ and a fixed cutoff function $\varphi \in C_0^\infty(\mathbb{R}, [0, 1])$, we have the following estimates

$$|\bar{\partial}R_j(z)| \le c_1|z|^{-1/2} + c_1|r'(|z|)| + c_1\varphi(|z|), \qquad z \in \Omega_i, \qquad i = 1,2,3,4, j = 1,2; \tag{3.22}$$

$$|\bar{\partial}R_j(z)| \le c_1|z - 1|, \qquad z \in \Omega_1, \Omega_4; \qquad |\bar{\partial}R_j(z)| \le c_1|z + 1|, \qquad z \in \Omega_2, \Omega_3. \tag{3.23}$$

Extending R by $R(z)\|_{z \in \Omega_j \cup \Omega_{j+1}} = R_j(z)$, such that the symmetry $R(z) = -\overline{R(\bar{z}^{-1})}$ holds.

Proof. We just prove the case for R_1 in $\overline{\Omega}_1$. The proof for other cases is just an analogue.

From (2.28) and (2.30), $z = \pm 1$ are singularities of the scattering coefficients $a(z)$ and $b(z)$, which implies that $z = 1$ is a singularity of $R_1(z)$. But one can eliminate this singularity by using $T(z)^{-2}$. From (2.22) and (3.5) we have

$$\frac{\bar{r}(z)}{1 - |r(z)|^2}T_+(z)^{-2} = \frac{\overline{b(z)}}{a(z)}\left(\frac{a(z)}{T_+(z)}\right)^2 \doteq \frac{\overline{J_b(z)}}{J_a(z)}\left(\frac{a(z)}{T_+(z)}\right)^2, \tag{3.24}$$

where

$$J_b(z) = \det[\psi_1^-(z; x, t), \psi_1^+(z; x, t)] \qquad J_a(z) = \det[\psi_1^+(z; x, t), \psi_2^-(z; x, t)]. \tag{3.25}$$

Since in the scattering problem, X is traceless, we can then derive that the determinants of the Jost functions $\psi_j^\pm(z; x, t)$, $j = 1, 2$, are independent of x. The analyticity of the denominator in the r.h.s. of (3.24) can be obtained owing to Propositions 2.1 and 3.1.

We then introduce the cutoff functions $\chi_0, \chi_1 \in C_0^\infty(\mathbb{R}, [0, 1])$ with small support near 0 and 1 respectively. For any sufficiently small $s \in \mathbb{R}$, $1 = \chi_0(s) = \chi_1(s + 1)$. Moreover, defining $\chi_1(s) = \chi(s^{-1})$ ensures the symmetry which will be useful in the following content. Then we can rewrite $R_1(z)$ in $\mathbb{R}$ as $R_1(z) = R_{11}(z) + R_{12}(z)$ satisfying

$$R_{11}(z) = (1 - \chi_1(z))\frac{\bar{r}(z)}{1 - |r(z)|^2}T_+(z)^{-2}, \qquad R_{12}(z) = \chi_1(z)\frac{\overline{J_b(z)}}{J_a(z)}\left(\frac{a(z)}{T_+(z)}\right)^2. \tag{3.26}$$

For a fixed small $\delta_0 > 0$, extending the function $R_{11}(z)$ and $R_{12}(z)$ by

$$R_{11}(z) = (1 - \chi_1(|z|))\frac{\bar{r}(|z|)}{1 - |r(|z|)|^2}T_+(z)^{-2}\cos(k\arg z), \tag{3.27}$$

$$R_{12}(z) = f(|z|)g(z)\cos(k\arg z) + \frac{i|z|}{k}\chi_0(\frac{\arg z}{\delta_0})f'(|z|)g(z)\sin(k\arg z), \tag{3.28}$$

where $f'(s)$ denotes the derivative of $f(s)$ and

$$k \doteq \frac{\pi}{2\theta_0}, \qquad g(z) \doteq \left(\frac{a(z)}{T(z)}\right)^2, \qquad f(s) \doteq \chi_1(s)\frac{\overline{J_b(s)}}{J_a(s)}.$$

Direct calculation shows that R_1 defined in this way satisfies the symmetry $R_1(s) = -\overline{R_1(\bar{s}^{-1})}$.

Now we give the estimates of the $\bar{\partial}$-derivatives of (3.27)–(3.28). For R_{11}, we have

$$\bar{\partial} R_{11}(z) = -\frac{\bar{\partial}\chi_1(|z|)}{T(z)^2}\frac{\overline{r(|z|)}\cos(k\arg z)}{1-|r(|z|)|^2} + \frac{1-\chi_1(|z|)}{T(z)^2}\bar{\partial}\left(\frac{\overline{r(|z|)}\cos(k\arg z)}{1-|r(|z|)|^2}\right). \tag{3.29}$$

Note that for the fixed constants C and c, $1-|r(z)|^2 > c > 0$ as $z \in \operatorname{supp}(1-\chi_1(|z|))$ and $|T(z)^{-2}| \le C$ as $z \in \Omega_1 \cap \operatorname{supp}(1-\chi_1(|z|))$. For $z = u + \mathrm{i}v = \rho e^{\mathrm{i}\alpha}$, we have $\bar{\partial} = \frac{1}{2}(\partial_u + \mathrm{i}\partial_v) = \frac{e^{\mathrm{i}\alpha}}{2}(\partial_\rho + \frac{\mathrm{i}}{\rho}\partial_\alpha)$. As $T(z)$ and $g(z)$ are analytic in Ω_1, it follows that

$$\left|\frac{\bar{\partial}\chi_1(|z|)}{T(z)^2}\frac{\overline{r(|z|)}\cos(k\arg z)}{1-|r(|z|)|^2}\right| = \left|\frac{\frac{1}{2}e^{\mathrm{i}\alpha}\chi_1'\bar{r}\cos(k\alpha)}{T(z)^2(1-|r(|z|)|^2)}\right| \le c_1\varphi(|z|), \tag{3.30}$$

for some $\varphi \in C_0^\infty(\mathbb{R},[0,1])$ with a small support near 1 and with $\varphi = 1$ on $\operatorname{supp}\chi_1$. Using $r(0) = 0$ and $r(z) \in H^1(\mathbb{R})$, we have $|r(|z|)| \le |z|^{1/2}\|r'\|_{L^2(\mathbb{R})}$, then for some fixed constants C_2 and C_3, we have

$$\begin{aligned}
&\left|\frac{1-\chi_1(|z|)}{T(z)^2}\bar{\partial}\left(\frac{\overline{r(|z|)}\cos(k\arg z)}{1-|r(|z|)|^2}\right)\right| \\
&= \left|\frac{1-\chi_1(|z|)}{T(z)^2(1-|r(|z|)|^2)}\right|\left|\frac{1}{2}e^{\mathrm{i}\alpha}(\bar{r}'\cos(k\alpha) - \mathrm{i}k\bar{r}|z|^{-1}\sin(k\alpha))(1-|r(|z|)|^2)\right. \\
&\left. + \frac{1}{2}e^{\mathrm{i}\alpha}(r'\bar{r} + \bar{r}'r)\bar{r}\cos(k\alpha)\right| \\
&\le C_2|r'(z)| + C_3\frac{|r(z)|}{|z|} \le C_2|r'(z)| + C_3|z|^{-1/2}.
\end{aligned}$$

So we get the estimation for $\bar{\partial}R_{11}(z)$

$$|\bar{\partial}R_{11}(z)| \le c_1\varphi(|z|) + c_2|r'(z)| + c_3|z|^{-1/2}.$$

Now we estimate $|\bar{\partial}R_{12}(z)|$. We have

$$\begin{aligned}
\bar{\partial}R_{12}(z) = \frac{1}{2}e^{\mathrm{i}\alpha}g(z)\Bigg[&f'cos(k\alpha)(1-\chi_0(\frac{\alpha}{\delta_0})) - \frac{\mathrm{i}kf(\rho)}{\rho}\sin(k\alpha) \\
&+\frac{\mathrm{i}}{k}(\rho f'(\rho))'\sin(k\alpha)\chi_0(\frac{\alpha}{\delta_0}) + \frac{\mathrm{i}}{k\delta_0}f'(\rho)\sin(k\alpha)\chi_0'(\frac{\alpha}{\delta_0})\Bigg],
\end{aligned}$$

in which $g(z)$ is bounded. So we can state that $|\bar{\partial}R_{12}(z)| \le c_4\varphi(|z|)$ for a $\varphi \in C_0^\infty[\mathbb{R},[0,1]]$ supported near 1 and for a constant c_4, thus yielding (3.22).

Eventually, as $z \to 1$, we have

$$|\bar{\partial}R_{12}(z)| \le [\sin(k\alpha) + (1-\chi_0(\alpha/\delta_0))] = \mathcal{O}(\alpha),$$

so (3.23) follows immediately. □

Now we define the modified factorization on $\mathbb{R}$ as $V^{(1)}(z) = \hat{B}(z)\hat{B}^{-\dagger}(z)$, where

$$\hat{B}(z) = \begin{pmatrix} 1 & 0 \\ R_2e^{-2\mathrm{i}t\theta} & 1 \end{pmatrix}, \qquad \hat{B}^{\dagger}(z) = \begin{pmatrix} 1 & R_1e^{2\mathrm{i}t\theta} \\ 0 & 1 \end{pmatrix}.$$

Using 3.5, we define $m^{(2)}(z)$ to open the lenses:

$$m^{(2)}(z) = \begin{cases} m^{(1)}(z)\hat{B}^{\dagger}(z), & z \in \Omega_1 \cup \Omega_2; \\ m^{(1)}(z)\hat{B}(z), & z \in \Omega_3 \cup \Omega_4; \\ m^{(1)}(z), & z \in \mathbb{C}\setminus\bar{\Omega}. \end{cases} \tag{3.31}$$

Let

$$\Sigma^{(2)} = \bigcup_{j\in H\setminus\Lambda}\left\{z \in \mathbb{C} : |z \pm z_j| = \rho \text{ or } |z \pm \bar{z}_j| = \rho\right\}. \tag{3.32}$$

Then $m^{(2)}(z)$ satisfies the following $\bar{\partial}$-RH problem.

RH Problem 3.2. *Find a 2×2 matrix-valued function $m^{(2)}(z) = m^{(2)}(z;x,t)$ such that*

1. *$m^{(2)}(z)$ is continuous in $\mathbb{C}\setminus(\Sigma^{(2)}\cup\{z_{j_0}\})$, with continuous boundary values $m_+^{(2)}(z)$ and $m_-^{(2)}(z)$ on $\Sigma^{(2)}$ from the left and right, respectively.*
2. *$m^{(2)}(z)$ has the following asymptotics:*

$$m^{(2)}(z) = I + \mathcal{O}(z^{-1}), \quad z\to\infty; \qquad zm^{(2)}(z) = \sigma_2 + \mathcal{O}(z), \quad z \to 0. \tag{3.33}$$

3. *$m^{(2)}(z)$ satisfies the following jump relation*

$$m_+^{(2)}(z) = m_-^{(2)}(z)V^{(2)}(z),$$

where

- *for $j \in \triangle\setminus\Lambda$,*

$$V^{(2)}(z) = \begin{cases} \begin{pmatrix} 1 & -\frac{(z-z_j)e^{2it\theta(z_j)}}{c_j}T^{-2}(z) \\ 0 & 1 \end{pmatrix}, & |z-z_j| = \rho, \\ \begin{pmatrix} 1 & \frac{(z+\bar{z}_j)e^{2it\theta(z_j)}}{\bar{c}_j}T^{-2}(z) \\ 0 & 1 \end{pmatrix}, & |z+\bar{z}_j| = \rho, \\ \begin{pmatrix} 1 & 0 \\ -\frac{(z-\bar{z}_j)e^{-2it\theta(\bar{z}_j)}}{\bar{c}_j}T^2(z) & 1 \end{pmatrix}, & |z-\bar{z}_j| = \rho, \\ \begin{pmatrix} 1 & 0 \\ \frac{(z+z_j)e^{-2it\theta(\bar{z}_j)}}{c_j}T^2(z) & 1 \end{pmatrix}, & |z+z_j| = \rho, \end{cases} \tag{3.34}$$

- *for $j \in \nabla\setminus\Lambda$,*

$$V^{(2)}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ -\frac{c_je^{-2it\theta(z_j)}}{z-z_j}T^2(z) & 1 \end{pmatrix}, & |z-z_j| = \rho, \\ \begin{pmatrix} 1 & 0 \\ \frac{\bar{c}_je^{-2it\theta(z_j)}}{z+\bar{z}_j}T^2(z) & 1 \end{pmatrix}, & |z+\bar{z}_j| = \rho, \\ \begin{pmatrix} 1 & -\frac{\bar{c}_je^{2it\theta(\bar{z}_j)}}{z-\bar{z}_j}T^{-2}(z) \\ 0 & 1 \end{pmatrix}, & |z-\bar{z}_j| = \rho, \\ \begin{pmatrix} 1 & \frac{c_je^{2it\theta(\bar{z}_j)}}{z+z_j}T^{-2}(z) \\ 0 & 1 \end{pmatrix}, & |z+z_j| = \rho. \end{cases} \tag{3.35}$$

4. *For $z \in \mathbb{C}\setminus(\Sigma^{(2)}\cup\{z_{j_0}\})$, we have:*

$$\bar{\partial}m^{(2)}(z) = m^{(2)}(z)W(z), \tag{3.36}$$

where

$$W(z) = \begin{cases} \bar{\partial}\hat{B}^{\dagger}(z) = \begin{pmatrix} 0 & \bar{\partial}R_1e^{2it\theta} \\ 0 & 0 \end{pmatrix}, & z \in \Omega_1\cup\Omega_2 \\ \bar{\partial}\hat{B}(z) = \begin{pmatrix} 0 & 0 \\ \bar{\partial}R_2e^{-2it\theta} & 0 \end{pmatrix}, & z \in \Omega_3\cup\Omega_4 \\ 0 & \text{elsewhere}. \end{cases} \tag{3.37}$$

5. *If $\Lambda = \emptyset$, then $m^{(2)}(z)$ is analytic in $\mathbb{C}\setminus(\bar{\Omega}\cup\Sigma^{(2)})$. If there exists $j_0 \in \{0,1,\ldots,N-1\}$ such that $|\operatorname{Re} z_{j_0} - \xi_0| \le \rho$, then $m^{(2)}(z)$ is meromorphic in $\mathbb{C}\setminus(\overline{\Omega}\cup\Sigma^{(2)})$ with four simple poles $\pm z_{j_0}$ and $\pm\bar{z}_{j_0}$, satisfying the following residue conditions:*

(a) If $j_0 \in\triangle$, denoting $C_{j_0} = c_{j_0}^{-1} T'(z_{j_0})^{-2}$, we have

$$\begin{aligned}
\operatorname{Res}_{z=z_{j_0}} m^{(2)}(z) &= \lim_{z\to z_{j_0}} m^{(2)}(z)\begin{pmatrix} 0 & C_{j_0} e^{2it\theta(z_{j_0})} \\ 0 & 0 \end{pmatrix}, \\
\operatorname{Res}_{z=-z_{j_0}} m^{(2)}(z) &= \lim_{z\to -z_{j_0}} m^{(2)}(z)\begin{pmatrix} 0 & 0 \\ -C_{j_0} e^{2it\theta(z_{j_0})} & 0 \end{pmatrix}, \\
\operatorname{Res}_{z=-\bar z_{j_0}} m^{(2)}(z) &= \lim_{z\to -\bar z_{j_0}} m^{(2)}(z)\begin{pmatrix} 0 & -\bar C_{j_0} e^{2it\theta(z_{j_0})} \\ 0 & 0 \end{pmatrix}, \\
\operatorname{Res}_{z=\bar z_{j_0}} m^{(2)}(z) &= \lim_{z\to \bar z_{j_0}} m^{(2)}(z)\begin{pmatrix} 0 & 0 \\ \bar C_{j_0} e^{2it\theta(z_{j_0})} & 0 \end{pmatrix}.
\end{aligned} \tag{3.38}$$

(b) If $j_0 \in \nabla$, denoting $C_{j_0} = c_{j_0} T(z_{j_0})^2$, we have

$$\begin{aligned}
\operatorname{Res}_{z=z_{j_0}} m^{(2)}(z) &= \lim_{z\to z_{j_0}} m^{(2)}(z)\begin{pmatrix} 0 & 0 \\ C_{j_0} e^{-2it\theta(z_{j_0})} & 0 \end{pmatrix}, \\
\operatorname{Res}_{z=-z_{j_0}} m^{(2)}(z) &= \lim_{z\to -z_{j_0}} m^{(2)}(z)\begin{pmatrix} 0 & -C_{j_0} e^{-2it\theta(z_{j_0})} \\ 0 & 0 \end{pmatrix}. \\
\operatorname{Res}_{z=\bar z_{j_0}} m^{(2)}(z) &= \lim_{z\to \bar z_{j_0}} m^{(2)}(z)\begin{pmatrix} 0 & \bar C_{j_0} e^{-2it\theta(z_{j_0})} \\ 0 & 0 \end{pmatrix}, \\
\operatorname{Res}_{z=-\bar z_{j_0}} m^{(2)}(z) &= \lim_{z\to -\bar z_{j_0}} m^{(2)}(z)\begin{pmatrix} 0 & 0 \\ -\bar C_{j_0} e^{-2it\theta(z_{j_0})} & 0 \end{pmatrix}.
\end{aligned} \tag{3.39}$$

4. The large-time asymptotic analysis

4.1. Asymptotics of N-soliton solution

We will neglect the $\bar\partial$ component of the solution, then the remaining is a new RH problem with zero $\bar\partial$-derivatives in Ω. After that, the small norm theory can be used to analyze the rest problem.

Proposition 4.1. *Let $m^{(sol)}(z)$ represent the new RH problem derived by excluding the $\bar\partial$ component of RH Problem 3.2. Specifically, $m^{(sol)}(z)$ is the solution to $\bar\partial$-RH Problem 3.2 with $W \equiv 0$. For scattering data $\left\{ r(z), \{z_j, c_j\}_{j=0}^{N-1} \right\}$ in RH Problem 3.2, $m^{(sol)}(z)$ is equivalent to RH Problem 2.1 with the modified reflectionless scattering data $\{0, \{z_j, \tilde c_j\}_{j=0}^{N-1}\}$. Here, the modified connection coefficients $\tilde c_j$ are determined by*

$$\tilde c_j = c_j(x,t) \exp\left(-\frac{1}{i\pi} \int_{\mathbb{R}} \log(1-|r(s)|^2)\left(\frac{1}{s-z_j} - \frac{1}{2s}\right) ds \right). \tag{4.1}$$

Proof. When $W \equiv 0$, the $\bar\partial$-RH problem for $m^{(sol)}(z)$ becomes a new RH problem with jump contour $\Sigma^{(2)}$. The next transformation aims to map each circle in $\Sigma^{(2)}$ back to the corresponding poles. This is done to ensure that $\tilde m(z)$ possesses simple poles at each $\pm z_k$ or $\pm \bar z_k$ in $\mathcal{Z}$. Additionally, the transformation reverses the triangularity induced by (3.4) and (3.13):

$$\tilde m(z) = \left[\prod_{k\in\triangle} (-|z_k|^2) \right]^{\sigma_3} m^{(sol)}(z) F(z) \left[\prod_{k\in\triangle} \frac{(z-z_k)(z+\bar z_k)}{(zz_k-1)(z\bar z_k+1)} \right]^{-\sigma_3}, \tag{4.2}$$

where

- for $j \in\triangle \setminus \Lambda$,

$$F(z) = \begin{cases} \begin{pmatrix} 1 & \frac{(z-z_j)e^{2it\theta(z_j)}}{c_j} T^{-2}(z) \\ 0 & 1 \end{pmatrix}, & |z-z_j| < \rho, \\ \begin{pmatrix} 1 & -\frac{(z+\bar z_j)e^{2it\theta(z_j)}}{\bar c_j} T^{-2}(z) \\ 0 & 1 \end{pmatrix}, & |z+\bar z_j| < \rho, \\ \begin{pmatrix} 1 & 0 \\ \frac{(z-\bar z_j)e^{-2it\theta(\bar z_j)}}{\bar c_j} T^{2}(z) & 1 \end{pmatrix}, & |z-\bar z_j| < \rho, \\ \begin{pmatrix} 1 & 0 \\ -\frac{(z+z_j)e^{-2it\theta(\bar z_j)}}{c_j} T^{2}(z) & 1 \end{pmatrix}, & |z+z_j| < \rho, \end{cases} \tag{4.3}$$

- for $j \in \nabla \setminus \Lambda$,

$$F(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \frac{c_j e^{-2it\theta(z_j)}}{z-z_j} T^{2}(z) & 1 \end{pmatrix}, & |z-z_j| < \rho, \\ \begin{pmatrix} 1 & 0 \\ -\frac{\bar c_j e^{-2it\theta(z_j)}}{z+\bar z_j} T^{2}(z) & 1 \end{pmatrix}, & |z+\bar z_j| < \rho, \\ \begin{pmatrix} 1 & \frac{\bar c_j e^{2it\theta(\bar z_j)}}{z-\bar z_j} T^{-2}(z) \\ 0 & 1 \end{pmatrix}, & |z-\bar z_j| < \rho, \\ \begin{pmatrix} 1 & -\frac{c_j e^{2it\theta(\bar z_j)}}{z+z_j} T^{-2}(z) \\ 0 & 1 \end{pmatrix}, & |z+z_j| < \rho. \end{cases} \tag{4.4}$$

- Elsewhere, $F(z) = I$.

Note that there are some remarkable properties for $\tilde m$: (i) $\tilde m$ keeps the normalization conditions as $z \to 0$ and $z \to \infty$; (ii) the jump around the simple poles no longer exists; (iii) $\tilde m$ satisfies the residue conditions (2.52), with the modified connection coefficients $\tilde c_j$. Therefore, we verified that $\tilde m$ is the solution of RH Problem 2.1 with the reflectionless scattering data $\{0, \{z_k, \tilde c_k\}_{k=0}^{N-1}\}$. The symmetry $r(s^{-1}) = -\overline{r(s)}$, $s \in \mathbb{R}$, gives the modified connection coefficients $\tilde c_j = z_j |\tilde c_j|$. In a word, $m^{(sol)}(z)$ is the solution of RH Problem 2.1, with a N-soliton, reflectionless, potential $\tilde q(x,t)$, and the discrete spectrum $\mathcal{Z}$, but with the modified connection coefficients $\tilde c_j$. □

Proposition 4.2. *Let $\xi = \frac{x}{t}$ and $j_0 = j_0(\xi) \in \{-1, 0, 1, \dots, N-1\}$. Assume that $m^{\Lambda}(z)$ solves RH Problem 3.2 with $W(z) \equiv 0$ and $V^{(2)} \equiv I$. Then there exists an unique solution $m^{\Lambda}(z)$ to the above RH problem as follows:*

- *if $j_0(\xi) = -1$, corresponding to $\Lambda = \emptyset$, then all the discrete eigenvalues $\pm z_j$ are away from the critical lines. Moreover,*

$$m^{\Lambda}(z) = I + \frac{\sigma_2}{z}; \tag{4.5}$$

- *if $j_0(\xi) \in \nabla$, then*

$$m^{\Lambda}(z) = I + \frac{\sigma_2}{z} + \begin{pmatrix} \frac{\alpha_{j_0}^{\nabla}(x,t)}{z-z_{j_0}} - \frac{\sigma\bar\alpha_{j_0}^{\nabla}(x,t)}{z+\bar z_{j_0}} & \frac{\bar\beta_{j_0}^{\nabla}(x,t)}{z-\bar z_{j_0}} - \frac{\sigma\beta_{j_0}^{\nabla}(x,t)}{z+z_{j_0}} \\ \frac{\beta_{j_0}^{\nabla}(x,t)}{z-z_{j_0}} - \frac{\sigma\bar\beta_{j_0}^{\nabla}(x,t)}{z+\bar z_{j_0}} & \frac{\bar\alpha_{j_0}^{\nabla}(x,t)}{z-\bar z_{j_0}} - \frac{\sigma\alpha_{j_0}^{\nabla}(x,t)}{z+z_{j_0}} \end{pmatrix},$$

$$\alpha_{j_0}^{\nabla}(x,t) = -iz_{j_0} \bar\beta_{j_0}^{\nabla},$$

$$\beta_{j_0}^{\nabla}(x,t)=\begin{cases}\dfrac{\sin\theta_{j_0}(1+\tanh\varphi_{j_0})(z_{j_0}\mathrm{sech}\varphi_{j_0}-\frac{1}{2}\cos\theta_{j_0}(1+\tanh\varphi_{j_0}))}{\left(\frac{1}{2}(1+\tanh\varphi_{j_0})\cos\theta_{j_0}-\sec\theta_{j_0}\mathrm{sech}\varphi_{j_0}\right)^2+\tan^2\theta_{j_0}\mathrm{sech}^2\varphi_{j_0}}, & \sigma=0\\ \dfrac{i(1+\tanh\varphi_{j_0})}{\mathrm{sech}\varphi_{j_0}-2(1+\tanh\varphi_{j_0})}, & \sigma=1\end{cases} \tag{4.6}$$

where $\theta_{j_0}=\arg z_{j_0}$ *and when* $\sigma=1$, $z_{j_0}\neq i$ *and when* $\sigma=0$, $z_{j_0}=i$.

- *if* $j_0(\xi)\in\triangle$, *then*

$$m^{\Lambda}(z)=I+\frac{\sigma_2}{z}+\begin{pmatrix}\frac{\alpha_{j_0}^{\triangle}(x,t)}{z-\bar{z}_{j_0}}-\frac{\bar{\alpha}_{j_0}^{\triangle}(x,t)}{z+z_{j_0}} & \frac{\bar{\beta}_{j_0}^{\triangle}(x,t)}{z-z_{j_0}}-\frac{\beta_{j_0}^{\triangle}(x,t)}{z+\bar{z}_{j_0}}\\ \frac{\beta_{j_0}^{\triangle}(x,t)}{z-\bar{z}_{j_0}}-\frac{\bar{\beta}_{j_0}^{\triangle}(x,t)}{z+z_{j_0}} & \frac{\bar{\alpha}_{j_0}^{\triangle}(x,t)}{z-z_{j_0}}-\frac{\alpha_{j_0}^{\triangle}(x,t)}{z+\bar{z}_{j_0}}\end{pmatrix},$$

$$\alpha_{j_0}^{\triangle}(x,t)=i\bar{z}_{j_0}\bar{\beta}_{j_0},$$

$$\beta_{j_0}^{\triangle}(x,t)=\frac{\sin\theta_{j_0}(1-\tanh\varphi_{j_0})(\bar{z}_{j_0}\mathrm{sech}\varphi_{j_0}-\frac{1}{2}\cos\theta_{j_0}(1-\tanh\varphi_{j_0}))}{\left(\frac{1}{2}(1-\tanh\varphi_{j_0})\cos\theta_{j_0}-\sec\theta_{j_0}\mathrm{sech}\varphi_{j_0}\right)^2+\tan^2\theta_{j_0}\mathrm{sech}^2\varphi_{j_0}}. \tag{4.7}$$

In case 2 *and* 3, *the real phase* φ_{j_0} *is given by*

$$\varphi_{j_0}=2\,\mathrm{Im}\,z_{j_0}(x+(4(\mathrm{Re}\,z_{j_0})^2+2)t+x_{j_0}), \tag{4.8}$$

$$x_{j_0}=\frac{1}{2\,\mathrm{Im}\,z_{j_0}}\left\{\log\left(\frac{|c_{j_0}|}{\mathrm{Im}\,z_{j_0}}\prod_{k\in\triangle,k\neq j_0}\left|\frac{(z_{j_0}-z_k)(z_{j_0}+\bar{z}_k)}{(z_{j_0}z_k-1)(z_{j_0}\bar{z}_k+1)}\right|\right)-\frac{\mathrm{Im}\,z_{j_0}}{\pi}\int_{\mathbb{R}}\frac{\log(1-|r(s)|^2)}{|s-z_{j_0}|^2}ds\right\}.$$

Moreover, as $z\to\infty$, *we have the following asymptotics for* $m^{\Lambda}(z)$

$$m^{\Lambda}(z)=I+\frac{1}{z}\begin{pmatrix}-i+\bar{\beta}_{j_0}-\sigma\beta_{j_0} & \\ i+\beta_{j_0}-\sigma\bar{\beta}_{j_0} & \bar{\alpha}_{j_0}-\sigma\alpha_{j_0}\end{pmatrix}+\mathcal{O}(z^{-2}), \tag{4.9}$$

from which we can get the soliton solution

$$sol(z_{j_0},x-x_{j_0},t)=\lim_{z\to\infty}izm_{21}^{\Lambda}(z)=-1+i(\beta_{j_0}-\sigma\bar{\beta}_{j_0})$$

$$=\begin{cases}-1-\dfrac{2\sin^2\theta_{j_0}\mathrm{sech}\varphi_{j_0}(1+\tanh\varphi_{j_0})}{\left(\frac{1}{2}(1-\tanh\varphi_{j_0})\cos\theta_{j_0}-\sec\theta_{j_0}\mathrm{sech}\varphi_{j_0}\right)^2+\tan^2\theta_{j_0}\mathrm{sech}^2\varphi_{j_0}}, & \text{as } j_0\in\nabla\\ -1-\dfrac{1+\tanh\varphi_{j_0}}{\mathrm{sech}\varphi_{j_0}-2(1+\tanh\varphi_{j_0})}, & \text{as } \sigma=1\\ -1+\dfrac{2\sin^2\theta_{j_0}\,sech\varphi_{j_0}(1-\tanh\varphi_{j_0})}{\left(\frac{1}{2}(1-\tanh\varphi_{j_0})\cos\theta_{j_0}-\sec\theta_{j_0}\mathrm{sech}\varphi_{j_0}\right)^2+\tan^2\theta_{j_0}\mathrm{sech}^2\varphi_{j_0}}, & \text{as } j_0\in\triangle\,.\end{cases} \tag{4.10}$$

Proof. Owing to $V\equiv I$ and $W\equiv 0$, we have that $m^{\Lambda}(z)$ is meromorphic with simple poles at $z=0$, $\pm z_{j_0}$ and $\pm\bar{z}_{j_0}$ (as $j_0\neq -1$). If $\Lambda=\emptyset$, then (4.5) can be obtained directly from the asymptotic behavior of RH Problem 3.2. If $\Lambda\neq\emptyset$, then there must be a $j_0\in\Lambda$, note that $C_0\doteq c_{j_0}T(z_{j_0})^2$ meets the condition $C_0=z_{j_0}|C_0|$ as $c_{j_0}=z_{j_0}|c_{j_0}|$. If $j_0\in\nabla$, then $m^{\Lambda}(z)$ coincides with the solution of RH Problem 2.1 with reflectionless scattering data, simple poles at 0, $\pm z_{j_0}$ and $\pm\bar{z}_{j_0}$, and with connection coefficient C_0. Then the symmetries (2.17) which is also satisfied by m^{Λ}, and the residue conditions (3.39) suggest that $\alpha_{j_0}^{\nabla}=-iz_{j_0}\bar{\beta}_{j_0}^{\nabla}$ and

$$m^{\Lambda}(z)=I+\frac{\sigma_2}{z}+\begin{pmatrix}\frac{\alpha_{j_0}}{z-z_{j_0}} & \frac{\bar{\beta}_{j_0}}{z-\bar{z}_{j_0}}\\ \frac{\beta_{j_0}}{z-z_{j_0}} & \frac{\bar{\alpha}_{j_0}}{z-\bar{z}_{j_0}}\end{pmatrix}+\begin{pmatrix}\frac{-\bar{\alpha}_{j_0}}{z+\bar{z}_{j_0}} & \frac{-\beta_{j_0}}{z+z_{j_0}}\\ \frac{-\bar{\beta}_{j_0}}{z+\bar{z}_{j_0}} & \frac{-\alpha_{j_0}}{z+z_{j_0}}\end{pmatrix}.$$

Under the notation $e^{\varphi_{j_0}}=\frac{|c_{j_0}|}{\mathrm{Im}\,z_{j_0}}e^{2\,\mathrm{Im}\,z_{j_0}(x+(4(\mathrm{Re}\,z_{j_0})^2+2)t)}$, using the residue conditions (3.39), we can easily obtain (4.6). If $j_0\in\triangle$, the calculation is similar, but with $\alpha_{j_0}^{\triangle}=i\bar{z}_{j_0}\bar{\beta}_{j_0}^{\triangle}$ and

$$m^{\Lambda}(z)=I+\frac{\sigma_2}{z}+\begin{pmatrix}\frac{\alpha_{j_0}}{z-\bar{z}_{j_0}} & \frac{\bar{\beta}_{j_0}}{z-z_{j_0}}\\ \frac{\beta_{j_0}}{z-\bar{z}_{j_0}} & \frac{\bar{\alpha}_{j_0}}{z-z_{j_0}}\end{pmatrix}+\begin{pmatrix}\frac{-\bar{\alpha}_{j_0}}{z+z_{j_0}} & \frac{-\beta_{j_0}}{z+\bar{z}_{j_0}}\\ \frac{-\bar{\beta}_{j_0}}{z+z_{j_0}} & \frac{-\alpha_{j_0}}{z+\bar{z}_{j_0}}\end{pmatrix}. \quad \square$$

4.2. Small norm RH problem and estimate of errors

In this section, we will analyze a small norm RH problem.

Proposition 4.3. *The jump matrix* $V^{(2)}(z)$ *has the following estimate*

$$\|V^{(2)}(z)-I\|_{L^p(\Sigma^{(2)})}\leq ce^{-2t\rho^2}. \tag{4.11}$$

Proof. For $|z-z_j|=\rho$ and $j\in\nabla\setminus\Lambda$, we have

$$\|V^{(2)}(z)-I\|_{L^{\infty}(\Sigma^{(2)})}=\left|-\frac{c_j}{z-z_j}T(z)^2e^{-2it\theta(z_j)}\right|\leq ce^{2t\,\mathrm{Im}\,z_j(\xi+4(\mathrm{Re}\,z_j)^2+2)}\leq ce^{-2t\rho^2}. \tag{4.12}$$

The others can be obtained in a similar manner. $\square$

Define

$$m^{(err)}(z)=m^{(sol)}(z)m^{\Lambda}(z)^{-1}, \tag{4.13}$$

then $m^{(err)}(z)$ satisfies the following RH problem:

RH Problem 4.1. *Find a* 2×2 *matrix-valued function* $m^{(err)}(z)$ *such that*

1. $m^{(err)}(z)$ *is analytic in* $\mathbb{C}\setminus\Sigma^{(2)}$, *where* $\Sigma^{(2)}$ *is defined in* (3.32).
2. $m^{(err)}(z)$ *has the following asymptotics:*
$$m^{(err)}(z)=I+\mathcal{O}(z^{-1}),\qquad z\to\infty. \tag{4.14}$$
3. $m^{(err)}(z)$ *satisfies the following jump relation:*
$$m_+^{(err)}(z)=m_-^{(err)}(z)V^{(err)}(z),\qquad z\in\Sigma^{(2)}, \tag{4.15}$$
where the jump matrix is defined as
$$V^{(err)}(z)=m^{\Lambda}(z)V^{(2)}(z)m^{\Lambda}(z)^{-1}, \tag{4.16}$$
and the jump contour is shown in Fig. 7.

The following proposition gives an estimate for the jump matrix $V^{(err)}$.

Proposition 4.4. *The jump matrix* $V^{(err)}(z)$ *in* (4.16) *satisfies*

$$\|V^{(err)}(z)-I\|_{L^p(\Sigma^{(2)})}\leq ce^{-2\rho^2t},\qquad 1\leq p\leq\infty. \tag{4.17}$$

Moreover, the solution of RH Problem 4.1 exists.

Proof. For $z\in\Sigma^{(2)}$, we have

$$|V^{(err)}(z)-I|=|m^{\Lambda}(z)(V^{(err)}(z)-I)m^{\Lambda}(z)^{-1}|\leq c|V^{(2)}-I|\leq ce^{-2\rho^2t}. \tag{4.18}$$

According to Beals–Coifman theory, the jump matrix $V^{(err)}(z)$ has the trivial decomposition

$$V^{(err)}(z)=(b_-)^{-1}b_+,\qquad b_-=I,\qquad b_+=V^{(err)}(z),$$

so we can define

$$(w_e)_-=I-b_-=0,\qquad (w_e)_+=b_+-I=V^{(err)}-I,$$
$$w_e=(w_e)_++(w_e)_-=V^{(err)}-I,$$

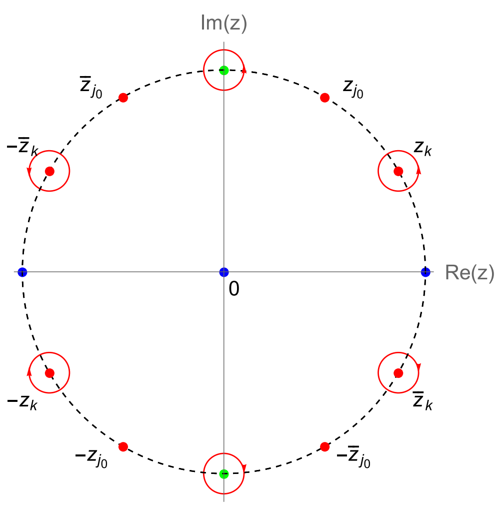


Fig. 7. The jump contour $\Sigma^{(2)}$ for $m^{(err)}(z)$ is the union of all circle $|z-z_j| = \rho$, $j \in H \setminus \{j_0\}$.

and

$$C_{w_e} f = C_-(f(w_e)_+) + C_+(f(w_e)_-) = C_-(f(V^{(err)} - I)), \tag{4.19}$$

where C_- is the Cauchy projection operator:

$$C_- f(z) = \lim_{z' \to z \in \Sigma^{(2)}} \frac{1}{2\pi \mathrm{i}} \int_{\Sigma^{(2)}} \frac{f(s)}{s - z'} ds,$$

and $\|C_-\|_{L^2}$ is bounded. Then the solution of RH Problem 4.1 can be expressed as

$$m^{(err)}(z) = I + \frac{1}{2\pi \mathrm{i}} \int_{\Sigma^{(2)}} \frac{\mu_e(s)(V^{(err)}(s) - I)}{s - z} ds, \tag{4.20}$$

where $\mu_e(z) \in L^2(\Sigma^{(2)})$ satisfies $(1 - C_{w_e})\mu_e(z) = I$. Using (4.18) and (4.19), we can obtain that

$$\|C_{w_e}\|_{L^2(\Sigma^{(2)})} \leq \|C_-\|_{L^2(\Sigma^{(2)})} \|V^{(err)}(z) - I\|_{L^\infty(\Sigma^{(2)})} \leq c e^{-2\rho^2 t}. \tag{4.21}$$

Hence, the resolvent operator $(1 - C_{w_e})^{-1}$ exists. Consequently, μ_e and the solution of RH Problem 4.1 $m^{(err)}(z)$ exist. □

Proposition 4.5. *Let $\xi = \frac{x}{t}$, then for any (x,t) in $\left\{(x,t) : -6 < \frac{x}{t} < -2\right\}$, as $t \gg 1$, uniformly for $z \in \mathbb{C}$ we have the estimate*

$$m^{(sol)}(z) = m^{\Lambda}(z)[I + \mathcal{O}(\mathrm{e}^{-2\rho^2 t})].$$

Especially, for z sufficiently large, we have the asymptotic extension

$$m^{(sol)}(z) = m^{\Lambda}(z)[I + z^{-1}\mathcal{O}(\mathrm{e}^{-2\rho^2 t}) + \mathcal{O}(z^{-2})]. \tag{4.22}$$

So we have the asymptotics of the potential

$$q^{(sol),N}(x,t) = q^{\Lambda}(x,t) + \mathcal{O}(\mathrm{e}^{-2\rho^2 t}) = sol(z_{j_0}, x - x_{j_0}, t) + 1 + \mathcal{O}(\mathrm{e}^{-2\rho^2 t}), \tag{4.23}$$

where

$$q^{(sol),N}(x,t) = \lim_{z\to\infty} izm_{21}^{(sol)}(z), \quad q^{\Lambda}(x,t) = \lim_{z\to\infty} izm_{21}^{\Lambda}(z).$$

Proof. From (4.20), we can split $m^{(err)}(z) - I$ into two parts

$$m^{(err)}(z) - I = \frac{1}{2\pi \mathrm{i}} \int_{\Sigma^{(2)}} \frac{V^{(err)}(s) - I}{s - z} ds + \frac{1}{2\pi \mathrm{i}} \int_{\Sigma^{(2)}} \frac{(\mu_e(s) - I)(V^{(err)}(s) - I)}{s - z} ds.$$

It follows that

$$\begin{aligned} |m^{(err)}(z) - I| &\leq \|V^{(err)}(s) - I\|_{L^2(\Sigma^{(2)})} \left\| \frac{1}{s - z} \right\|_{L^2(\Sigma^{(2)})} \\ &\quad + \|V^{(err)}(s) - I\|_{L^\infty(\Sigma^{(2)})} \|\mu_e(s) - I\|_{L^2(\Sigma^{(2)})} \left\| \frac{1}{s - z} \right\|_{L^2(\Sigma^{(2)})} \\ &\leq c e^{-2\rho^2 t}. \end{aligned}$$

Therefore, we have the estimate for $m^{(sol)}(z)$

$$m^{(sol)}(z) = m^{\Lambda}(z) \left[I + \frac{1}{2\pi i} \int_{\Sigma^{(2)}} \frac{\mu_e(s)(V^{(err)}(s) - I)}{s - z} ds \right] = m^{\Lambda}(z) \left[I + \mathcal{O}(\mathrm{e}^{-2\rho^2 t}) \right].$$

As $z \to \infty$, $m^{(err)}(z)$ has the following asymptotic extension

$$m^{(err)}(z) = I + \frac{m_1^{(err)}}{z} + \mathcal{O}(z^{-1}), \tag{4.24}$$

where

$$m_1^{(err)} = -\frac{1}{2\pi \mathrm{i}} \int_{\Sigma^{(2)}} \mu_e(s)(V^{(err)}(s) - I) ds$$

$$= -\frac{1}{2\pi \mathrm{i}} \int_{\Sigma^{(2)}} (V^{(err)}(s) - I) ds - \frac{1}{2\pi \mathrm{i}} \int_{\Sigma^{(2)}} (\mu_e(s) - I)(V^{(err)}(s) - I) ds, \tag{4.25}$$

from which we have the following estimate

$$m_1^{(err)}(z) \leq \|V^{(err)}(z) - I\|_{L^1} + \|\mu_e(z) - I\|_{L^2} \|V^{(err)}(z) - I\|_{L^2} \leq c e^{-2\rho^2 t}. \quad \square$$

4.3. Analysis on a pure $\bar{\partial}$-problem

In this section, we use $m^{(sol)}(z)$ to reduce $m^{(2)}(z)$ to a pure $\bar{\partial}$-problem and analyze it. Define the function

$$m^{(3)}(z) = m^{(2)}(z) \left(m^{(sol)}(z) \right)^{-1}, \tag{4.26}$$

then $m^{(3)}(z)$ satisfies the following $\bar{\partial}$-problem.

RH Problem 4.2. *Find a 2×2 matrix-valued function $m^{(3)}(z)$ such that*

1. *$m^{(3)}(z)$ is analytic in $\mathbb{C} \setminus \overline{\Omega}$, and continuous in $\mathbb{C}$.*
2. *$m^{(3)}(z)$ has the asymptotics:*

$$m^{(3)}(z) = I + \mathcal{O}(z^{-1}), \qquad z \to \infty. \tag{4.27}$$

3. *For $z \in \mathbb{C}$, we have*

$$\bar{\partial} m^{(3)}(z) = m^{(3)}(z) W^{(3)}(z), \tag{4.28}$$

where $W^{(3)}(z) = m^{(sol)}(z) W(z) \left(m^{(sol)}(z) \right)^{-1}$ and $W(z)$ is defined in (3.37).

Proof. The definition (4.26) implies that $m^{(3)}(z)$ has no jump on the circles $|z \pm z_j| = \rho$ nor $|z \pm \bar{z}_j| = \rho$ since

$$\left(m_-^{(3)}(z) \right)^{-1} m_+^{(3)}(z) = m_-^{(sol)}(z) V^{(2)}(z) \left(m_+^{(sol)}(z) \right)^{-1} = I.$$

The asymptotics (4.27) and $\bar{\partial}$-derivative (4.28) can be obtained directly from those of $m^{(2)}(z)$ and $m^{(sol)}(z)$. We just need to prove that $m^{(3)}$ defined in (4.26) has no isolated singularities. At $z = 0$ we have $\left(m^{(sol)}(z) \right)^{-1} = (1 - z^{-2})^{-1} \left(m^{(sol)}(z) \right)^T$, then

$$\lim_{z\to 0} m^{(3)}(z) = \lim_{z\to 0} \frac{\left(z m^{(2)}(z) \right) z \left(m^{(sol)}(z) \right)^T}{z^2 - 1} = I, \tag{4.29}$$

which implies that $z = 0$ is not a singularity of $m^{(3)}(z)$. $\det m^{(sol)}(z) = 1 - z^{-2}$ indicates that $z = \pm 1$ might be the potential singularities. Applying the symmetries (2.49) to the expansion of $m^{(2)}$ and $m^{(sol)}$, we get for some constants c_1 and c_2,

$$m^{(2)}(z) = \begin{pmatrix} c_1 & \mp \mathrm{i} c_1 \\ \pm \mathrm{i} \bar{c}_1 & \bar{c}_1 \end{pmatrix} + \mathcal{O}(z \mp 1),$$

$$\left(m^{(sol)}(z) \right)^{-1} = \frac{\pm 1}{2(z \mp 1)} \begin{pmatrix} c_2 & \pm \mathrm{i} \bar{c}_2 \\ \mp \mathrm{i} c_2 & \bar{c}_2 \end{pmatrix}^T + \mathcal{O}(1).$$

It then follows that

$$\lim_{z\to\pm1} m^{(3)}(z) = \mathcal{O}(1), \tag{4.30}$$

which implies that $m^{(3)}(z)$ has no singularities at $z = \pm1$. Now we are going to prove that z_{j_0} is not a singularity of $m^{(3)}$. If $m^{(2)}(z)$ has pole at z_{j_0}, $j_0 \in \nabla \cap \Lambda$, then we have the residue condition

$$\mathrm{Res}_{z=z_{j_0}}\, m^{(2)}(z) = \lim_{z\to z_{j_0}} m^{(2)}(z)\mathcal{N}_{j_0},$$

where

$$\mathcal{N}_{j_0} = \begin{pmatrix} 0 & 0 \\ C_{j_0} e^{2it\theta(z_{j_0})} & 0 \end{pmatrix},$$

so $m^{(2)}$ has the Laurent expansion

$$m^{(2)}(z) = \frac{\mathrm{Res}_{z=z_{j_0}}\, m^{(2)}}{z - z_{j_0}} c(z_{j_0}) + \mathcal{O}(z - z_{j_0}),$$

where $c(z_{j_0})$ is a constant matrix. It follows immediately that

$$\mathrm{Res}_{z=z_{j_0}}\, m^{(2)}(z) = c(z_{j_0})\mathcal{N}_{j_0},$$

which gives another form of expansion

$$m^{(2)}(z) = c(z_{j_0}) \left[I + \frac{\mathcal{N}_{j_0}}{z - z_{j_0}} \right] + \mathcal{O}(z - z_{j_0}). \tag{4.31}$$

Since $m^{(sol)}$ and $m^{(2)}$ have the same residue conditions, owing to $\det m^{(2)}(z) = \det m^{(sol)}(z) = 1 - z^{-2}$, we have

$$\left(m^{(sol)}(z_{j_0})\right)^{-1} = \frac{z_{j_0}^2}{z_{j_0}^2 - 1} \left[I - \frac{\mathcal{N}_{j_0}}{z - z_{j_0}} \right] c(z_{j_0})^T + \mathcal{O}(z - z_{j_0}). \tag{4.32}$$

Taking the above into (4.26), we have

$$m^{(3)}(z) = \frac{z_{j_0}^2}{z_{j_0}^2 - 1} c(z_{j_0}) \left[I + \frac{\mathcal{N}_{j_0}}{z - z_{j_0}} \right] \left[I - \frac{\mathcal{N}_{j_0}}{z - z_{j_0}} \right] c(z_{j_0})^T + \mathcal{O}(1), \tag{4.33}$$

from which we can state that $m^{(3)}(z)$ is bounded near the pole and the pole is removable. We then give the $\bar\partial$ derivative of $m^{(3)}$

$$\begin{aligned}\bar\partial m^{(3)}(z) &= \bar\partial m^{(2)} \left(m^{(sol)}(z)\right)^{-1} = m^{(2)}(z)\bar\partial R^{(2)} \left(m^{(sol)}(z)\right)^{-1} \\ &\left[m^{(2)}(z)\left(m^{(sol)}(z)\right)^{-1}\right]\left[m^{(sol)}(z)\bar\partial R^{(2)}\left(m^{(sol)}(z)\right)^{-1}\right] = m^{(3)}(z)W^{(3)}.\end{aligned}$$

where $W^{(3)}(z) = m^{(sol)}(z)W(z)\left(m^{(sol)}(z)\right)^{-1}$. □

The solution of the pure $\bar\partial$ problem can be expressed as

$$m^{(3)}(z) = I - \frac{1}{\pi} \int\int_{\mathbb{C}} \frac{m^{(3)}(s)W^{(3)}(s)}{s - z} dA(s) \tag{4.34}$$

where $dA(s)$ is the Lebesgue measure in $\mathbb{R}$. And it can also be expressed by operator equation

$$(I - J)m^{(3)}(z) = I \Longleftrightarrow m^{(3)}(z) = I + Jm^{(3)}(z),$$

where J is the Cauchy operator

$$Jf(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{f(s)W^{(3)}(s)}{s - z} dA(s) = \frac{1}{\pi z} * f(z)W^{(3)}(z). \tag{4.35}$$

Then we will show that J is small-norm for large t.

Proposition 4.6. *We have J: $L^\infty(\mathbb{C}) \to L^\infty(\mathbb{C}) \cap C^0(\mathbb{C})$ and for fixed $\xi_0 \in (0, 2)$ there exists a constant $C = C(q_0, \xi_0)$, such that for all $t \gg 1$ and for all $|\xi + 4| \le \xi_0$,*

$$\|J\|_{L^\infty(\mathbb{C})\to L^\infty(\mathbb{C})} \le Ct^{-1/2}. \tag{4.36}$$

Proof. Here we just consider the case when $f(z) \in L^\infty(\Omega_1)$. From (4.35), we have

$$|JF(z)| \le c\|f\|_{L^\infty(\mathbb{C})} \iint_{\mathbb{C}} \frac{|W^{(3)}(s)|}{|s - z|} dA(s),$$

where

$$|W^{(3)}(s)| \le |m^{(sol)}(s)|^2 \left|1 - s^{-2}\right|^{-1} |W(s)|.$$

For $z \in \Omega_1$, there exists a C_1 such that

$$\|m^{(sol)}\| \le C_1(1 + |s|^{-1}) = C_1|s|^{-1}\langle s\rangle.$$

Then for a fixed constant c_1, we have

$$|W^{(3)}(s)| \le c_1\langle s\rangle|s - 1|^{-1}|\bar\partial R_1(s)|e^{\mathrm{Re}(2it\theta(s))}. \tag{4.37}$$

Since $\frac{\langle s\rangle}{|1+s|} = \mathcal{O}(1)$ in Ω_1, we have the estimate

$$|Jf(z)| \le c_1 \iint_{\Omega_1} \frac{\langle s\rangle|\bar\partial R_1(s)|e^{\mathrm{Re}(2it\theta(s))}}{|s - z||s - 1|} dA(s) \doteq c_1(I_1 + I_2 + I_3),$$

where

$$I_1 = \iint_{\Omega_1} \frac{\langle s\rangle|\bar\partial R_1(s)|e^{\mathrm{Re}(2it\theta(s))}\chi_{[0,1)}}{|s - z||s - 1|} dA(s),$$

$$I_2 = \iint_{\Omega_1} \frac{\langle s\rangle|\bar\partial R_1(s)|e^{\mathrm{Re}(2it\theta(s))}\chi_{[1,2)}}{|s - z||s - 1|} dA(s),$$

$$I_3 = \iint_{\Omega_1} \frac{\langle s\rangle|\bar\partial R_1(s)|e^{\mathrm{Re}(2it\theta(s))}\chi_{[2,\infty)}}{|s - z||s - 1|} dA(s),$$

where $\chi_{[0,1)}(|s|) + \chi_{[1,2)}(|s|) + \chi_{[2,\infty)}(|s|)$ is the partition of unity.

We first estimate I_3. Since $\langle s\rangle|s - 1|^{-1} \le \kappa$ for $|s| \ge 2$, for a fixed κ, we only need to prove that

$$I_3 \le \kappa \iint_{\Omega_1} \frac{(c_2|s|^{-1/2} + c_2|r'(s)| + c_2\varphi(|s|))e^{\mathrm{Re}(2it\theta)}\chi_{[1,\infty)}(|s|)}{|s - z|} dA(s) \le ct^{1/2}, \tag{4.38}$$

for some fixed c_2 and c. Let $s = u + iv$, since $|\xi + 4| \le \xi_0$ for $\xi_0 \in (0, 2)$, we have $\mathrm{Re}(2it\theta(s)) \le -c'tv$. Let $z = z_R + iz_I$, $1/q + 1/p = 1$ and $p > 2$. For the integrals in (4.38) including $f(|s|) = |r'(s)|$ or $f(|s|) = \varphi(|s|)$, we can denote

$$\begin{aligned} I_{31} &\doteq \int_0^\infty e^{-c'tv} dv \int_v^\infty \frac{f(|s|)\chi_{[1,\infty)}(|s|)}{|s - z|} du \\ &\le \int_0^\infty e^{-c'tv} \|f(|s|)\|_{L^2(v,\infty)} \left\| \frac{\chi_{[1,\infty)}(|s|)}{s - z} \right\|_{L^2(v,\infty)} dv. \end{aligned} \tag{4.39}$$

While

$$\begin{aligned} \left\| \frac{\chi_{[1,\infty)}(|s|)}{s - z} \right\|^2_{L^2(v,\infty)} &\le \int_v^\infty \frac{1}{|s - z|^2} du \le \int_{-\infty}^{+\infty} \frac{1}{(u - z_R)^2 + (v - z_I)^2} du \\ &\xlongequal{y=\frac{u - z_R}{v - z_I}} \frac{1}{|v - z_I|} \int_{-\infty}^{+\infty} \frac{1}{1 + y^2} dy = \frac{\pi}{|v - z_I|}, \end{aligned}$$

and

$$\begin{aligned} \|f(|s|)\|^2_{L^2(v,\infty)} &= \int_v^\infty \left|f(\sqrt{u^2 + v^2})\right|^2 du \xlongequal{\tau=\sqrt{u^2+v^2}} \int_{\sqrt{2}v}^\infty |f(\tau)|^2 \frac{\sqrt{u^2 + v^2}}{u} d\tau \\ &\le \sqrt{2} \int_{\sqrt{2}v}^\infty |f(\tau)|^2 d\tau \le \|f(s)\|^2_{L^2(\mathbb{R})}. \end{aligned} \tag{4.40}$$

Putting the above two estimates into (4.39), we then have

$$I_{31} \le c\|f\|^2_{L^2(\mathbb{R})} \left[\int_0^{z_I} \frac{e^{-c'tv}}{\sqrt{z_I - v}} dv + \int_{z_I}^\infty \frac{e^{-c'tv}}{\sqrt{v - z_I}} dv \right]. \tag{4.41}$$

Using the inequality $\sqrt{z_I}\mathrm{e}^{-c'tz_Iw} \le ct^{-1/2}w^{-1/2}$, we can estimate the r.h.s in (4.41)

$$\int_0^{z_I}\frac{\mathrm{e}^{-c'tv}}{\sqrt{z_I-v}}dv \overset{w=\frac{v}{z_I}}{=\!=\!=} \int_0^1\frac{\sqrt{z_I}\mathrm{e}^{-c'tz_Iw}}{\sqrt{1-w}}dw \le ct^{-1/2}\int_0^1\frac{1}{\sqrt{w(1-w)}}dw \le ct^{-1/2},$$

and

$$\int_{z_I}^{\infty}\frac{\mathrm{e}^{-c'tv}}{\sqrt{v-z_I}}dv \le \int_0^{\infty}\frac{\mathrm{e}^{-c'tw}}{\sqrt{w}}dw = t^{-1/2}\int_0^{\infty}\frac{\mathrm{e}^{-c'\lambda}}{\sqrt{\lambda}}d\lambda \le ct^{-1/2}.$$

Subsequently, we have

$$I_{31} \le ct^{-1/2}\|f\|_{L^2(\mathbb{R})}. \tag{4.42}$$

Then we estimate the terms in (4.38) including $f(|s|) = |s|^{-1}$. Denote

$$\begin{aligned} I_{32} &\doteq \int_0^{\infty}\mathrm{e}^{-c'tv}dv\int_0^{\infty}\frac{\chi_{[1,\infty)}(|s|)|s|^{-1/2}}{|s-z|}du \\ &\le \int_0^{\infty}\mathrm{e}^{-c'tv}\||s|^{-1/2}\|_{L^p(v,\infty)}\||s-z|^{-1}\|_{L^q(v,\infty)}dv. \end{aligned} \tag{4.43}$$

Then for $p > 2$, we have

$$\||s|^{-1/2}\|_{L^p(v,\infty)} = v^{1/p-1/2}(\int_1^{\infty}\frac{1}{(1+x)^{p/4}}dx)^{1/p} \le cv^{1/p-1/2},$$

similarly, we have

$$\||s-z|^{-1}\|_{L^q(v,\infty)} \le c|v-z_I|^{1/q-1}, \quad \text{where } \frac{1}{q}+\frac{1}{p}=1. \tag{4.44}$$

Putting the above two estimates into (4.43) we have

$$I_{32} \le c\left[\int_0^{z_I}\mathrm{e}^{-c'tv}v^{1/p-1/2}|v-z_I|^{1/q-1}dv + \int_{z_I}^{\infty}\mathrm{e}^{-c'tv}v^{1/p-1/2}|v-z_I|^{1/q-1}dv\right]. \tag{4.45}$$

Then we give the estimate for the r.h.s in (4.45)

$$\int_0^{z_I}\mathrm{e}^{-c'tv}v^{1/p-1/2}|v-z_I|^{1/q-1}dv \overset{w=v/z_I}{=\!=\!=} \int_0^1 \sqrt{z_I}\mathrm{e}^{-c'tz_Iw}w^{1/p-1/2}|1-w|^{1/q-1}dw \le ct^{-1/2},$$

and

$$\int_{z_I}^{\infty}\mathrm{e}^{-c'tv}v^{1/p-1/2}|v-z_i|^{1/q-1}dv \overset{v=z_I+w}{=\!=\!=} \int_0^{\infty}\mathrm{e}^{-c't(z_I+w)}(z_I+w)^{1/p-1/2}w^{1/q-1}dw$$
$$\le_0^{\infty}\mathrm{e}^{-c'tw}w^{-1/2}dw = t^{-1/2}\int_0^{\infty}y^{-1/2}\mathrm{e}^{-y}dy \le ct^{-1/2}.$$

Plugging the above two estimations into (4.45), we have

$$I_{32} \le ct^{-1/2}. \tag{4.46}$$

So far we have proved (4.38). Now we are going to estimate I_2. According to (3.23), for $|s| \le 2$, we have $|\bar\partial R_j(s)| \le c_1|s-1|$ and $\langle s\rangle < \sqrt{5}$, so we have

$$I_2 \le \sqrt{5}c_1\iint_{\Omega_1}\frac{\mathrm{e}^{-\mathrm{Re}(2it\theta)}\chi_{[1,2)}(|s|)}{|s-z|}dA(s). \tag{4.47}$$

It follows immediately from the estimate of I_3 that $I_2 \le ct^{-1/2}$. Finally, we give the estimation of I_1. Let $w = 1/\bar z$ and $\tau = 1/\bar s$, then we have

$$\begin{aligned} I_1 &= \iint_{\Omega_1}\frac{|\partial_\tau\bar R_1|\mathrm{e}^{\mathrm{Re}(2it\theta(|\tau|))}\chi_{[1,\infty)}(|\tau|)}{|\tau^{-1}-w^{-1}||\tau^{-1}-1||\tau|^4}\left|\frac{\partial\tau}{\partial\bar s}\right|dA(\tau) \\ &= |w|\iint_{\Omega_1}\frac{|\bar\partial R_1|\mathrm{e}^{\mathrm{Re}(2it\theta(|\tau|))}\chi_{[1,\infty)}(|\tau|)}{|\tau-w||\tau-1|}dA(\tau). \end{aligned} \tag{4.48}$$

If $|w| \le 3$, it is obvious that the estimate of I_1 becomes that of I_2. While if $|w| \ge 3$, we have

$$\begin{aligned} I_1 \le{}& 3\iint_{|\tau|\ge\frac{|w|}{2}}\frac{|\bar\partial R_1|\mathrm{e}^{\mathrm{Re}(2it\theta(|\tau|))}\chi_{[1,\infty)}(|\tau|)}{|\tau-w|}dA(\tau) \\ &+ 2\iint_{1\le|\tau|\le\frac{|w|}{2}}\frac{|\bar\partial R_1|\mathrm{e}^{\mathrm{Re}(2it\theta(|\tau|))}\chi_{[1,\infty)}(|\tau|)}{|\tau-1|}dA(\tau). \end{aligned}$$

It can be estimated by the same method as before, so that $I_1 \le ct^{-1/2}$. Then we have proved (4.36). □

We now show that the equation

$$m^{(3)} = I + Jm^{(3)}$$

holds in the distributional sense. In fact, for test function $\phi \in C_0^{\infty}(\mathbb{C},\mathbb{C})$, the differential equation

$$\bar\partial\phi(z) = f(z) \tag{4.49}$$

has a solution

$$\phi(z) = \frac{1}{\pi z} * f(z) = \frac{1}{\pi}\iint_{\mathbb{C}}\frac{f(w)}{z-w}dA(w).$$

Using (4.28) and (4.35), we have

$$\begin{aligned} \iint_{\mathbb{C}}\bar\partial m^{(3)}(w)\phi(w)dA(w) &= \iint_{\mathbb{C}} m^{(3)}(w)W^{(3)}(w)\left[\frac{1}{\pi}\iint_{\mathbb{C}}\frac{\bar\partial\phi(z)}{z-w}dA(z)\right]dA(w) \\ &= -\iint_{\mathbb{C}} Jm^{(3)}(z)\bar\partial\phi(z)dA(z) = \iint_{\mathbb{C}}\bar\partial[Jm^{(3)}(z)]\phi(z)dA(z), \end{aligned}$$

where we use the fact that the order of integration can be exchanged since $\frac{m^{(3)}(w)W^{(3)}(w)\bar\partial\phi(z)}{w-z} \in L^1(\mathbb{C})$. Therefore, in the distributional sense, we have $\bar\partial[m^{(3)} - Jm^{(3)}] = 0$, which means that $m^{(3)}(z) = I + Jm^{(3)}(z)$. Now we expand $m^{(3)}(z;x,t)$ as follows

$$m^{(3)}(z;x,t) = I + \frac{m_1^{(3)}(x,t)}{z} + \mathcal{O}(z^{-1}), \tag{4.50}$$

where

$$m_1^{(3)}(x,t) = \frac{1}{\pi}\iint_{\mathbb{C}} m^{(3)}(s)W^{(3)}(s)dA(s). \tag{4.51}$$

The following proposition gives the estimate for $m_1^{(3)}(x,t)$.

Proposition 4.7. *For $-6 < \xi < -2$, there exist constants t_1 and c such that $m_1^{(3)}(x,t)$ satisfies:*

$$|m_1^{(3)}(x,t)| \le ct^{-1} \quad \text{for} \quad |x/t+4| < 2 \quad \text{and} \quad t \ge t_1. \tag{4.52}$$

Proof. Proposition 4.6 implies that for $t \gg 1$ and $|\xi+4| < \xi_0$, we have $\|m^{(3)}\|_{L^\infty} \le c$. Using (4.37) and (4.51), we have

$$|m_1^{(3)}(x,t)| \le c\iint_{\Omega_1}\frac{\langle s\rangle|\bar\partial R_1|\mathrm{e}^{\mathrm{Re}(2it\theta)}}{s-1}dA(s) \le c(I_1+I_2+I_3), \tag{4.53}$$

where

$$I_1 = \iint_{\Omega_1}\frac{\langle s\rangle|\bar\partial R_1|\mathrm{e}^{\mathrm{Re}(2it\theta)}\chi_{[0,1)}(|s|)}{s-1}dA(s),$$

$$I_2 = \iint_{\Omega_1}\frac{\langle s\rangle|\bar\partial R_1|\mathrm{e}^{\mathrm{Re}(2it\theta)}\chi_{[1,2)}(|s|)}{s-1}dA(s),$$

$$I_3 = \iint_{\Omega_1}\frac{\langle s\rangle|\bar\partial R_1|\mathrm{e}^{\mathrm{Re}(2it\theta)}\chi_{[2,\infty)}(|s|)}{s-1}dA(s).$$

For $|s| \ge 2$, using $\langle s\rangle|s-1|^{-1} = \mathcal{O}(1)$, and fixing a $p > 2$, $q \in (1,2)$, we have

$$\begin{aligned} I_3 &\le c\iint_{\Omega_1}[|r'(|z|)| + \varphi(|s|) + |z|^{-1/2}]\mathrm{e}^{\mathrm{Re}(2it\theta)}\chi_{[1,\infty)}(|s|)dA(s) \\ &\le c\int_0^{\infty}\|\mathrm{e}^{-c'tv}\|_{L^2(\max\{v,\frac{1}{\sqrt{2}}\},\infty)}dv \\ &\quad + c\int_0^{\infty}\|\mathrm{e}^{-c'tv}\|_{L^2(\max\{v,\frac{1}{\sqrt{2}}\},\infty)}\||z|^{-1/2}\|_{L^q(v,\infty)}dv \\ &\le c\int_0^{\infty}\mathrm{e}^{-c'tv}(t^{-1/2}v^{-1/2} + t^{-1/p}v^{-1/p+1/q-1/2})dv \le ct^{-1}. \end{aligned}$$

For $s\in[0,2]$, we have $\langle s\rangle\le\sqrt{5}$. Applying an approach similar to the estimation of I_3, we derive the inequality $I_2\le ct^{-1}$, where the difference lies in substituting $|r'(|z|)|+\varphi(|s|)$ with the function $f=\chi_{[1,2]}(|s|)$. For $s\in[0,1]$, variable transformations $w=\bar z^{-1}$ and $r=\bar s^{-1}$ give that

$$I_1=\iint_{\Omega_1}\mathrm{e}^{\mathrm{Re}(2\mathrm{i}t\theta(w))}|\bar\partial R_1||w-1|^{-1}\chi_{[1,\infty)}(|w|)|w|^{-1}dA(s)\le ct^{-1}.$$

So we have proved the estimate. □

5. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. For sufficiently large $z\in\mathbb{C}\setminus\overline{\Omega}$, we have

$$m(z)=T(\infty)^{\sigma_3}m^{(3)}(z)m^{(sol)}(z)T(\infty)^{-\sigma_3}\left[I-\frac{1}{z}T_1^{-\sigma_3}+\mathcal{O}(z^{-2})\right],\tag{5.1}$$

where

$$T_1=\sum_{k\in\vartriangle}4\mathrm{i}\,\mathrm{Im}\,z_k-\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\log(1-|r(s)|^2)ds.$$

Below we discuss the asymptotic behavior of $q(x,t)$ as t large under different conditions.

1. If $\Lambda=\emptyset$, from (4.5), (4.22) and (4.50), we have

$$m(z)=I+\frac{1}{z}T(\infty)^{\sigma_3}\left[\sigma_2+m_1^{(3)}(x,t)-T_1^{-\sigma_3}+\mathcal{O}(\mathrm{e}^{-2\rho^2t})\right]T(\infty)^{-\sigma_3}.\tag{5.2}$$

Using the reconstruction formula, we have the following asymptotics for $q(x,t)$

$$q(x,t)=-1+\mathcal{O}(t^{-1}).\tag{5.3}$$

2. If $\Lambda\neq\emptyset$, we have

$$m^{(sol)}(z)=I+\frac{m_1^{(sol)}(x,t)}{z}+\mathcal{O}(z^{-2}).$$

Plugging the above and (4.50) into (5.1), we have

$$m(z)=I+\frac{1}{z}T(\infty)^{\sigma_3}[m_1^{(3)}(x,t)+m_1^{(sol)}(x,t)-T_1^{\sigma_3}]T(\infty)^{-\sigma_3}.$$

Again, using the reconstruction formula, we then have

$$q(x,t)=\lim_{z\to\infty}\mathrm{i}zm_{21}(z)=T(\infty)^{-2}q^{(sol),N}(x,t)+\mathcal{O}(t^{-1}).\tag{5.4}$$

Note that $|z_k|=1$ and $\bar z_k^{-1}=z_k$. Then (3.7) gives us $T(\infty)^{-2}=1$, which directly leads to the asymptotic stability

$$|q(x,t)-q^{(sol),N}(x,t)|\le ct^{-1}.\tag{5.5}$$

Next we show that the N-Soliton solutions for the mKdV equation have the property of soliton resolution. Consider the order (1.4), and the sets$\nabla=\{j:j\ge j_0\}$, $\vartriangle=\{j:j<j_0\}$, and $\Lambda=\emptyset$ or $\{j_0\}$. We could rewrite the asymptotics of $q^{(sol),N}$ (4.23) in terms of $q^{\Lambda}(x,t)$

$$q^{(sol),N}(x,t)=q^{\Lambda}(x,t)+\mathcal{O}(e^{-2\rho^2t}),$$

which combining with (5.5) gives

$$q(x,t)-q^{\Lambda}(x,t)=\left(q(x,t)-q^{(sol),N}(x,t)\right)+\left(q^{(sol),N}(x,t)-q^{\Lambda}(x,t)\right)=\mathcal{O}(t^{-1}).$$

Again by (4.23), we have

$$q(x,t)=[sol(z_j,x-x_j,t)+1]+\mathcal{O}(t^{-1}),\quad j=0,\dots,N-1.\tag{5.6}$$

By (5.3) and (5.6), we get soliton resolution of the defocusing mKdV equation

$$q(x,t)=-1+\sum_{j=0}^{N-1}[sol(z_j,x-x_j,t)+1]+\mathcal{O}(t^{-1}).\quad\square\tag{5.7}$$

Proof of Theorem 1.2. For q_0 close enough to $q^{(sol),M}(x,0)$, using the Lipschitz continuity (2.6)–(2.8) in Proposition 2.1, we can immediately get the relation of poles in (1.9). Then applying Theorem 1.1 to q_0, we can obtain (1.7). Finally, simple calculations (1.7) give (1.10). □

CRediT authorship contribution statement

Zechuan Zhang: Writing – original draft. **Taiyang Xu:** Writing – review & editing. **Engui Fan:** Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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