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On the Cauchy problem of defocusing mKdV equation with finite density initial data: Long time asymptotics in soliton-less regions

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Abstract

We investigate the long time asymptotics for the solutions to the Cauchy problem of defocusing modified Kortweg-de Vries (mKdV) equation with finite density initial data. The present paper is the subsequent work of our previous paper [arXiv:2108.03650], which gives the soliton resolution for the defocusing mKdV equation in the central asymptotic sector $\{(x,t) : |\xi| < 6\}$ with $\xi := x/t$. In the present paper, via the Riemann-Hilbert (RH) problem associated to the Cauchy problem, the long-time asymptotics in the soliton-less regions $\{(x,t) : |\xi| > 6, |\xi| = \mathcal{O}(1)\}$ for the defocusing mKdV equation are further obtained. It is shown that the leading term of the asymptotics is in compatible with the "background solution" and the error terms are derived via rigorous analysis.

MSC: 35Q51; 35Q15; 35C20; 37K15; 37K40

Keywords: Defocusing mKdV equation; Riemann-Hilbert problem; $\bar{\partial}$ steepest descent method; Long time asymptotics; Soliton-less regions

Contents

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1. Introduction

In the present work, we investigate the long time asymptotics in soliton-less regions for the defocusing modified Kortweg-de Vries (mKdV) equation with finite density initial data:

$$q_t(x,t) - 6q^2(x,t)q_x(x,t) + q_{xxx}(x,t) = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \tag{1.1}$$

$$q(x,0) = q_0(x), \quad \lim_{x \to \pm\infty} q_0(x) = \pm 1. \tag{1.2}$$

Remark 1.1. Generally, the finite type initial data is presented by the nonzero boundary condition $\lim_{x\to\pm\infty} q_0(x) = q_\pm$, $|q_\pm| = q_0$. Taking the following transformation

$$u = q/q_0, \quad \tilde{x} = q_0 x, \quad \tilde{t} = q_0^3 t,$$

we have

$$u_{\tilde{t}}(\tilde{x},\tilde{t})-6u^2(\tilde{x},\tilde{t})u_{\tilde{x}}(\tilde{x},\tilde{t})+u_{\tilde{x}\tilde{x}\tilde{x}}(\tilde{x},\tilde{t})=0,$$

with the normalized boundary conditions $\lim_{\tilde{x}\to\pm\infty} u = q_\pm/q_0$, $|q_\pm/q_0| = 1$. Based on the analysis, we directly choose the boundary condition of initial data as (1.2) for convenience.

Remark 1.2. The "soliton-less regions" doesn't represent that there exist no solitons in our present work. Indeed, we use a interpolation transformation to convert residue conditions for poles to jump conditions such that the jump matrices vanish as $t\to\infty$. In our result, the solitons make few contributions (exponential decay) for the obtained asymptotics.

The mKdV equation arises in various of physical fields, such as acoustic wave and phonons in a certain anharmonic lattice [28,33], Alfvén wave in a cold collision-free plasma [18,22]. A considerable amount of work has been carried out around the long time asymptotics for defocusing mKdV equation (1.1). The earliest work can be traced back to Segur and Ablowitz [2], who extend the method developed by Zakharov and Manakov [34] to derive the leading asymptotics for the solution of the mKdV equation, including full information on the phase. The most influential work to investigate the long time behavior of integrable PDEs is the nonlinear steepest descent method which was firstly proposed by Deift and Zhou (Deift-Zhou method) to study the defocusing mKdV equation [14]. Lenells proves a nonlinear steepest descent theorem for RH problems with Carleson jump contours, where jump matrices admit low regularity and slow decay [23]. Recently, Chen and Liu extend the asymptotics to the solution for defocusing mKdV equation with initial data in lower regularity spaces [11]. The works mentioned above refer to that initial data $q_0(x)$ admits zero boundary conditions (ZBCs, i.e., $q_0(x)\to 0$ as $x\to\pm\infty$).

Studies for the long time asymptotic behavior of the integrable systems with nonzero boundary conditions (NBCs) have been investigated in a number previous articles. Specifically, the nonzero boundary conditions could be divided into the asymmetric NBCs (i.e., $q_0(x)\to q_\pm$ with $|q_+|\neq|q_-|$, also be called "step-like" initial data) and symmetric NBCs (i.e., $q_0(x)\to q_\pm$ with $|q_+|=|q_-|\neq 0$). For the long time asymptotic behavior for integrable PDEs with asymmetric NBCs, refer [5,6,9,19–21,25]. For the symmetric NBCs, a lot of works for long time asymptotics have been investigated around nonlinear Schrödinger (NLS) equation, see [7,8,30,31]. S. Cuccagna and R. Jenkins [10] develop the $\bar{\partial}$ generalization which was firstly proposed by McLaughlin and his collaborators [4,12,26,27] to verify the soliton resolution for defocusing NLS equation with finite initial data in an asymptotic soliton regime $|x/2t|<1$. The method used in [10] is applied to investigate the asymptotics for $|x/2t|>1$ by Wang and Fan [32].

For the defocusing mKdV equation with finite density initial data defined by (1.1)-(1.2), Zhang and Yan [36] use the inverse scattering transform (IST) to express the solution in terms of the associated RH problem and prove that the discrete spectrum are distributed on the unit circle in the complex plane. For comparison, only focusing mKdV equation posses discrete spectrum under ZBCs. In the presence of discrete spectrum for defocusing mKdV equation with finite density initial data, we exhibit the soliton resolution and asymptotic stability in the previous article [35] for $|\xi|<6$, and the asymptotics for $|\xi|>6$, $|\xi|=\mathcal{O}(1)$ in the present work.

1.1. Main results

The main result of this work is exhibited in the following theorem that reveals the long time asymptotic behavior of the solution $q(x,t)$ of defocusing mKdV equation (1.1) in different asymptotic sectors (see Fig. 1), where

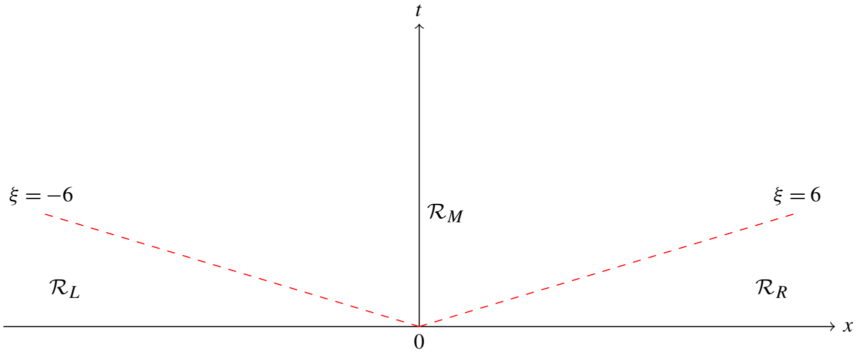


Fig. 1. Asymptotic sectors for the solution $q(x,t)$ for mKdV equation.

$$\mathcal{R}_L=\{(x,t):\xi<-6,|\xi|=\mathcal{O}(1)\},\quad \mathcal{R}_M=\{(x,t):-6<\xi<6\},$$
$$\mathcal{R}_R=\{(x,t):\xi>6,|\xi|=\mathcal{O}(1)\},\ \ \xi:=x/t.$$

Theorem 1.3. *Let $q(x,t)$ be the solution for the Cauchy problem* (1.1) *with generic data $q_0(x)-\tanh(x)\in H^{4,4}(\mathbb{R})$ associated to scattering data $\left\{r(z),\{\eta_n,c_n\}_{n=1}^{2N}\right\}$. As $t\to+\infty$, the following three asymptotics are shown.*

(a) *For $(x,t)\in\mathcal{R}_L$ (left field),*

$$q(x,t)=-1+t^{-\frac{1}{2}}f(\xi)+\mathcal{O}(t^{-\frac{3}{4}}),\tag{1.3}$$

where

$$f(\xi):=\sum_{j=1}^{4}\epsilon_j\left(2\epsilon_j\theta''(\xi_j)\right)^{-\frac{1}{2}}\left(1-\xi_j^{-2}\right)^{-1}\cdot\left(\beta_{12}^{(\xi_j)}-\frac{1}{\xi_j^2}\beta_{21}^{(\xi_j)}\right),$$

with $\epsilon_j=(-1)^{j+1}$ for $j=1,2,3,4$, ξ_j defined by (3.8)-(3.9) *for $j=1,2,3,4$, $\beta_{12}^{(\xi_j)}$ and $\beta_{21}^{(\xi_j)}$ defined by* (A.24)-(A.26) *for $j=1,3$, while by* (A.50)-(A.52) *for $j=2,4$.*

(b) *For $(x,t)\in\mathcal{R}_M$*

$$q(x,t)=-1+\sum_{j=0}^{N}[sol(z_j,x-x_j,t)+1]+\mathcal{O}(t^{-1}).\tag{1.4}$$

(c) *For $(x,t)\in\mathcal{R}_R$ (right field),*

$$q(x,t)=1+\mathcal{O}(t^{-1}).\tag{1.5}$$

Remark 1.4. Comparing to the results in [10], [32] for defocusing NLS equation, the asymptotics in Theorem 1.3 is real-valued, which owes to the mKdV equation is a real-valued integrable PDE. It's find that the asymptotics (1.3) is formally similar to the asymptotics in [32], and the sub-leading term stems the contribution from four saddle points (in the case of mKdV equation) and two saddle points (in the case of NLS equation) respectively. The other difference between the present work and [32] is the asymptotics in right field, where the error bounds of the former mainly stem from the $\bar{\partial}$ estimation by $\mathcal{O}(t^{-1})$.

Remark 1.5. $q^{sol}(x,t) = \tanh(x)$ is the stationary solution of (1.1)-(1.2), which is called the dark soliton.

Remark 1.6. $|\xi| = \mathcal{O}(1)$ is needed to ensure the following issues:
- The saddle points ξ_1, ξ_4 defined by (3.8) are bounded;
- The estimates for $\Im\theta(z)$, jump matrix and $\bar{\partial}$ derivatives are reasonable, see Proposition 4.4, Proposition 4.7, Proposition 4.9, Proposition 4.14, Proposition 4.15, Proposition 4.28 and Proposition 5.3;
- The higher power term of the expansion for $\theta(z)$ near saddle points could decay as $t \to \infty$, see Remark 4.19.

If removing the condition $|x/t| = \mathcal{O}(1)$, we can turn to study the large x asymptotic behavior in a similar way to large t asymptotics.

Remark 1.7. In an early version of this paper, $\mathcal{R}_R$ is set by $\xi \in (-2, +\infty)$, $|\xi| = \mathcal{O}(1)$ instead of $\xi \in (6, +\infty)$, $|\xi| = \mathcal{O}(1)$. The reason why we modify this condition is that we find when $\xi \in (-2, 6)$, the asymptotics Theorem 1.3(b) is matched with the asymptotics in the right field by setting the index $\Lambda = \emptyset$ in [35]. Indeed, solitons make no contribution for the asymptotics when $\xi > -2$. On one side, we can set $\xi \in (-2, 6)$ as the special part of $\mathcal{R}_R$ (but we should know where $\Lambda = \emptyset$). On the other side, we can set $\xi \in (-2, 6)$ as the part of right field, that's because there are no soliton contributions for the asymptotics.

Remark 1.8. The smoothness and decay properties of the reflection coefficient $r(z)$ are needed in our analysis.
- Proposition 2.5 shows that: $q_0 - \tanh(x) \in H^{3,3}(\mathbb{R}) \Rightarrow q_0 - \tanh(x) \in L^{1,2}(\mathbb{R}) \Rightarrow r(z) \in H^1(\mathbb{R})$.
- Eq. (2.29) shows $r(z) = \mathcal{O}(z^{-2})$ as $z \to \infty$, and we can obtain that $r(z)$ also belongs to $L^{2,1}(\mathbb{R})$. Moreover, $r(z) \in H^{1,1}(\mathbb{R}) = L^{2,1}(\mathbb{R}) \cap H^1(\mathbb{R})$. It's Corollary 2.7.

The condition $q_0 - \tanh(x) \in H^{4,4}(\mathbb{R})$ in Theorem 1.3 is needed to include all conditions to show that $r(z) \in H^1(\mathbb{R})$, which can help us bound the $\bar{\partial}$ derivatives of our extensions in Proposition 4.9, Proposition 4.28 and Proposition 5.3, etc.

1.2. Outline of this paper

The structure of this work is as follows.

Section 2 and Section 3 are the preliminary parts. In Section 2, we review the elementary results on the associated RH problem formulation of the Cauchy problem for the defocusing mKdV equation (1.1), which is the basis to analyze the asymptotic behavior of the defocusing mKdV equation in our work. In Section 3, we present the distribution of phase points of $\theta(z)$ and depict the signature tables of $e^{2it\theta}$ by some numerical figures.

In Section 4, we mainly deal with the asymptotics for $\xi \in \mathcal{R}_L$. In Subsection 4.1, jump matrix factorizations corresponding to this case are given. In Subsection 4.2, a scalar function $\delta(z)$ which could use the factorizations of the jump matrix along the real axis to deform the contours onto those which the oscillatory jump on the real axis for exponential decay, and a interpolation function $G(z)$ which interpolates the poles by trading them for jumps along small closed circles around each poles are introduced to make the first transformation $M(z) \to M^{(1)}(z)$. In Subsection 4.3, we open $\bar\partial$ lenses to set up a mixed $\bar\partial$-RH problem $M^{(2)}(z)$, which consists of a pure RH problem $M^{(PR)}(z)$ and a pure $\bar\partial$-problem $M^{(3)}(z)$. In Subsection 4.4, analysis on pure RH problem $M^{(PR)}(z)$ is exhibited, which refer to two standard parts: global parametrix $M^{(\infty)}(z)$ and local parametrix $M^{(LC)}(z)$. Error analysis via using small-norm RH theory is also given. In Subsection 4.5, we give the rigorous analysis for the pure $\bar\partial$-problem $M^{(3)}(z)$. In Subsection 4.6, the asymptotics in Theorem 1.3(a) is given by reviewing a series of transformations we use in this section.

Similar techniques to Section 4 are used to analyze the asymptotics for $\xi \in \mathcal{R}_R$ in Section 5.

1.3. Notations

We conclude this section with some notations used throughout this paper.

- Japanese bracket $\langle x\rangle := \sqrt{1+|x|^2}$ is widely used in some normed space.
- A weighted $L^{p,s}(\mathbb{R})$ is defined by $L^{p,s}(\mathbb{R}) = \{u \in L^p(\mathbb{R}) : \langle x\rangle^s u(x) \in L^p(\mathbb{R})\}$, with $\|u\|_{L^{p,s}(\mathbb{R})} := \|\langle x\rangle^s u\|_{L^p(\mathbb{R})}$.
- A Sobolev space is defined by $W^{m,p}(\mathbb{R}) = \{u \in L^p(\mathbb{R}) : \partial^j u(x) \in L^p(\mathbb{R})\ \text{ for }\ j = 0, 1, 2, \dots, m\}$, with $\|u\|_{W^{m,p}(\mathbb{R})} := \sum_{j=0}^m \|\partial^j u\|_{L^p(\mathbb{R})}$. Usually, we are used to expressing $H^m(\mathbb{R}) := W^{m,2}(\mathbb{R})$.
- A weighted Sobolev space is defined by $H^{m,s}(\mathbb{R}) := L^{2,s}(\mathbb{R}) \cap H^m(\mathbb{R})$.
- $\sigma_1, \sigma_2, \sigma_3$ are classical Pauli matrices as follows

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- $a \lesssim b$ i.e., $\exists c$, s.t. $a \leqslant cb$; $\epsilon_j := (-1)^{j+1}$.
- $\Re$ and $\Im$ represent real part and imaginary part of a complex variable respectively.

2. Direct and inverse scattering transform

2.1. Lax pair and spectral analysis

The defocusing mKdV equation (1.1) admits the following Lax pair [1]

$$\Phi_x = X\Phi, \quad \Phi_t = T\Phi, \tag{2.1}$$

where

$$X = ik\sigma_3 + Q, \quad T = 4k^2 X - 2ik\sigma_3(Q_x - Q^2) + 2Q^3 - Q_{xx}, \quad Q = \begin{pmatrix} 0 & q(x,t) \\ q(x,t) & 0 \end{pmatrix},$$

and $k \in \mathbb{C}$ is a spectral parameter.

By using the boundary condition of (1.2), the Lax pair (2.1) admits the following approximation

$$\Phi_{\pm,x} \sim X_\pm \Phi_\pm, \quad \Phi_{\pm,t} \sim T_\pm \Phi_\pm, \quad x \to \pm\infty, \tag{2.2}$$

where

$$X_\pm = ik\sigma_3 + Q_\pm, \quad T_\pm = (4k^2+2)X_\pm,$$

with $Q_\pm = \pm\sigma_1$.

The eigenvalues of $X_\pm$ are $\pm i\lambda$, which satisfy the equality

$$\lambda^2 = k^2 - 1. \tag{2.3}$$

Since λ is multi-valued, we introduce the following uniformization variable to ensure that our discussion is based on a complex plane rather than a Riemann surface

$$z = k + \lambda, \tag{2.4}$$

and obtain two single-valued functions

$$\lambda(z) = \frac{1}{2}(z - \frac{1}{z}), \quad k(z) = \frac{1}{2}(z + \frac{1}{z}). \tag{2.5}$$

Remark 2.1. In the present work, we use the uniformization technique. Indeed, the other techniques, such as [7,8] can be taken, which should deal with a branch cut. However, we have to pay more attention to singular points which appears via uniformization method.

Define two domains D_+, D_- and their boundary Σ on z-plane by

$$D_+ := \{z \in \mathbb{C}, \Im\lambda(1 + \frac{1}{|z|^2}) > 0\}, \ D_- := \{z \in \mathbb{C}, \Im\lambda(1 + \frac{1}{|z|^2}) < 0\}, \ \Sigma := \mathbb{R}\backslash\{0\}.$$

The “background solution” of the asymptotic spectral problem (2.2) is given by

$$\Phi_\pm \sim E_\pm(z)e^{i\lambda(z)x\sigma_3}, \tag{2.6}$$

where

$$E_\pm = \begin{pmatrix} 1 & \pm\frac{i}{z} \\ \mp\frac{i}{z} & 1 \end{pmatrix}.$$

Introducing the modified Jost solution

$$\mu_\pm = \Phi_\pm e^{-i\lambda(z)x\sigma_3}, \tag{2.7}$$

then we have

$$\mu_\pm \sim E_\pm, \quad \text{as} \quad x \to \pm\infty,$$

$$\det(\Phi_\pm) = \det(\mu_\pm) = \det(E_\pm) = 1 - \frac{1}{z^2}.$$

$\mu_\pm$ are defined by the Volterra type integral equations

$$\mu_\pm(x;z) = E_\pm(z) + \int_{\pm\infty}^{x} E_\pm(z) e^{i\lambda(z)(x-y)\hat{\sigma}_3}\left[(E_\pm^{-1}(z)\Delta Q_\pm(y)\,\mu_\pm(y;z)\right]dy, \quad z \neq \pm 1, \tag{2.8}$$

$$\mu_\pm(x;z) = E_\pm(z) + \int_{\pm\infty}^{x} \left[I + (x-y)\left(Q_\pm \pm i\sigma_3\right)\right]\Delta Q_\pm(y)\,\mu_\pm(y;z)\,dy, \quad z = \pm 1, \tag{2.9}$$

where $\Delta Q = Q - Q_\pm$.

The properties of $\mu_\pm$ are concluded in the following proposition, of which proof is similar to [10, Lemma 3.3] owing to defocusing mKdV equation and NLS equation admit the same spatial spectrum problem ($t = 0$).

Proposition 2.2. *Given $n \in \mathbb{N}_0$, let $q - \tanh(x) \in L^{1,n+1}(\mathbb{R})$, $q' \in W^{1,1}(\mathbb{R})$. Denote $\mu_{\pm,j}$ the j-th column of $\mu_\pm$.*
- For $z \in \mathbb{C}\setminus\{0\}$, $\mu_{+,1}(x,t;z)$ and $\mu_{-,2}(x,t;z)$ can be analytically extended to $\mathbb{C}_+$ and continuously extended to $\mathbb{C}_+ \cup \Sigma$; $\mu_{-,1}(x,t;z)$ and $\mu_{+,2}(x,t;z)$ can be analytically extended to $\mathbb{C}_-$ and continuously extended to $\mathbb{C}_- \cup \Sigma$.
- (Symmetry for $\mu_\pm$) $\mu_\pm(z) = \sigma_1\overline{\mu_\pm(\bar{z})}\sigma_1 = \overline{\mu_\pm(-\bar{z})}$, $\mu_\pm(z) = \frac{\pm 1}{z}\mu_\pm(z^{-1})\sigma_2$.
- (Asymptotic behavior of $\mu_\pm$ as $z \to \infty$) For $\Im z \geq 0$ as $z \to \infty$,

$$\mu_{+,1}(z) = e_1 + \frac{1}{z}\begin{pmatrix} -i\int_x^\infty (q^2-1)dx \\ -iq \end{pmatrix} + \mathcal{O}(z^{-2}), \tag{2.10}$$

$$\mu_{-,2}(z) = e_2 + \frac{1}{z}\begin{pmatrix} iq \\ i\int_x^\infty (q^2-1)dx \end{pmatrix} + \mathcal{O}(z^{-2}), \tag{2.11}$$

for $\Im z \leq 0$, as $z \to \infty$

$$\mu_{-,1}(z) = e_1 + \frac{1}{z}\begin{pmatrix} -i\int_x^\infty (q^2-1)dx \\ -iq \end{pmatrix} + \mathcal{O}(z^{-2}), \tag{2.12}$$

$$\mu_{+,2}(z) = e_2 + \frac{1}{z}\begin{pmatrix} iq \\ i\int_x^\infty (q^2-1)dx \end{pmatrix} + \mathcal{O}(z^{-2}). \tag{2.13}$$

- (Asymptotic behavior of $\mu_\pm$ as $z \to 0$) For $z \in \mathbb{C}_+$, as $z \to 0$,

$$\mu_{+,1}(z) = -\frac{i}{z}e_2 + \mathcal{O}(1), \qquad \mu_{-,2}(z) = -\frac{i}{z}e_1 + \mathcal{O}(1); \tag{2.14}$$

for $z \in \mathbb{C}_-$, as $z \to 0$,

$$\mu_{-,1}(z) = \frac{i}{z}e_2 + \mathcal{O}(1), \qquad \mu_{+,2}(z) = \frac{i}{z}e_1 + \mathcal{O}(1); \tag{2.15}$$

where $e_1 = (1,0)^{\mathrm{T}}$, $e_2 = (0,1)^{\mathrm{T}}$.

Since $\Phi_+(z)$ and $\Phi_-(z)$ are two fundamental matrix-valued solutions of (2.2) for $z \in \Sigma \backslash \{\pm 1\}$, thus there exists a scattering $S(z)$ such that

$$\Phi_+(x,t;z) = \Phi_-(x,t;z)S(z). \tag{2.16}$$

Owing to (2.7) and the symmetries of $\mu_\pm(z)$, $\Phi_\pm$ admit the following symmetry.

$$\Phi_\pm(z) = \sigma_1\overline{\Phi_\pm(\bar{z})}\sigma_1 = \overline{\Phi_\pm(-\bar{z})}, \quad \Phi_\pm(z) = \frac{\mp 1}{z}\Phi_\pm(z^{-1})\sigma_2. \tag{2.17}$$

Then the symmetry $S(z) = \sigma_1\overline{S(\bar{z})}\sigma_1 = \overline{S(-\bar{z})} = -\sigma_2 S(z^{-1})\sigma_2$ follows immediately. And $S(z)$ is given by

$$S(z) = \begin{pmatrix} a(z) & \overline{b(z)} \\ b(z) & \overline{a(z)} \end{pmatrix}, \quad z \in \Sigma \backslash \{\pm 1\}, \tag{2.18}$$

where $a(z)$, $b(z)$ are called scattering data.

Define

$$r(z) := \frac{b(z)}{a(z)}, \quad \tilde{r}(z) := \frac{b(\bar{z})}{a(\bar{z})}. \tag{2.19}$$

Several properties of $a(z)$, $b(z)$ and $r(z)$ are given as follows.

Proposition 2.3. *Let $z \in \Sigma \backslash \{\pm 1\}$, $a(z)$, $b(z)$ and $r(z)$ be the data as mentioned above.*
- The scattering coefficients can be expressed in terms of the Jost functions by

$$a(z) = \frac{\det(\Phi_{+,1}, \Phi_{-,2})}{1 - z^{-2}}, \quad b(z) = \frac{\det(\Phi_{-,1}, \Phi_{+,1})}{1 - z^{-2}}. \tag{2.20}$$

- For each $z \in \Sigma \backslash \{\pm 1\}$, we have

$$\det S(z) = |a(z)|^2 - |b(z)|^2 = 1, \quad |r(z)|^2 = 1 - |a(z)|^{-2} < 1. \tag{2.21}$$

- $a(z)$, $b(z)$ and the reflection coefficient $r(z)$ satisfy the symmetries

$$a(z) = \overline{a(-\bar{z})} = -\overline{a(\bar{z}^{-1})}, \tag{2.22}$$

$$b(z) = \overline{b(-\bar{z})} = \overline{b(\bar{z}^{-1})}, \tag{2.23}$$

$$r(z) = \overline{r(-\bar{z})} = -\overline{r(\bar{z}^{-1})}. \tag{2.24}$$

- The scattering data admit the asymptotics

$$\lim_{z \to \infty}(a(z) - 1)z = i\int_{\mathbb{R}}(q^2 - 1)dx, \ z \in \overline{\mathbb{C}_+}, \tag{2.25}$$

$$\lim_{z \to 0}(a(z) + 1)z^{-1} = i\int_{\mathbb{R}}(q^2 - 1)dx, \ z \in \overline{\mathbb{C}_+}, \tag{2.26}$$

and

$$|b(z)| = \mathcal{O}(|z|^{-2}), \qquad \text{as } |z| \to \infty, \ \ z \in \mathbb{R}, \tag{2.27}$$

$$|b(z)| = \mathcal{O}(|z|^{2}), \qquad \text{as } |z| \to 0, \ z \in \mathbb{R}. \tag{2.28}$$

So that

$$r(z) \sim z^{-2}, \quad |z| \to \infty; \qquad r(z) \sim z^2, \quad |z| \to 0. \tag{2.29}$$

Proof. The first item follows by applying Cramer's rule to (2.16). The second item can be obtained by direct calculation. The third item follows from the symmetry of $S(z)$. The fourth item follows from the first item and the asymptotics of $\mu_\pm$ in Proposition 2.2. □

Though $a(z)$ and $b(z)$ have singularities at ± 1, the reflection coefficient $r(z)$ remains bounded at $z = \pm 1$ with $|r(\pm 1)| = 1$. Indeed, as $z \to \pm 1$

$$a(z) = \frac{\pm S_\pm}{z \mp 1} + \mathcal{O}(1), \quad b(z) = \frac{\mp S_\pm}{z \mp 1} + \mathcal{O}(1) \tag{2.30}$$

where $S_\pm = \frac{1}{2}\det[\mu_{+.1}(\pm 1, x), \mu_{-,2}(\pm 1, x)]$. Then $\lim_{z \to \pm 1} r(z) = \mp i$ follows.

Remark 2.4. The above discussions suggest that scattering data exhibit singular behavior for z at $\pm 1, 0$. The singularities of these functions at $z = \pm 1$ can be removable, however, the singular behavior at $z = 0$ plays a non-trivial and unavoidable role in our analysis.

The next proposition shows that, given data q_0 with sufficient smoothness and decay properties, the reflection coefficients will also be smooth and decaying.

Proposition 2.5. *For given* $q - \tanh(x) \in L^{1,2}(\mathbb{R})$, $q' \in W^{1,1}(\mathbb{R})$, *then* $r(z) \in H^1(\mathbb{R})$.

Proof. The proof is the same with [35, Proposition 3.2]. □

Remark 2.6. $\|r\|_{H^1(\mathbb{R})}$ is widely used in the estimation below, such as Proposition 4.1, Proposition 4.9, Proposition 4.28, Proposition 5.3, etc. In fact, we can claim that $r(z) \in H^{1,1}(\mathbb{R})$, see the following corollary.

Corollary 2.7. *For given* $q - \tanh(x) \in L^{1,2}(\mathbb{R})$, $q' \in W^{1,1}(\mathbb{R})$, *we have* $r(z) \in H^{1,1}(\mathbb{R})$.

Proof. Since $H^{1,1}(\mathbb{R}) = L^{2,1}(\mathbb{R}) \cap H^1(\mathbb{R})$, what we need to prove is that $r \in L^{2,1}(\mathbb{R})$. With (2.29), we can see that

$$|z|^2 r^2(z) \sim |z|^{-2}, \quad |z| \to \infty \tag{2.31}$$

Thus

$$\int_{\mathbb{R}} |\langle z \rangle r(z)|^2 < \infty, \tag{2.32}$$

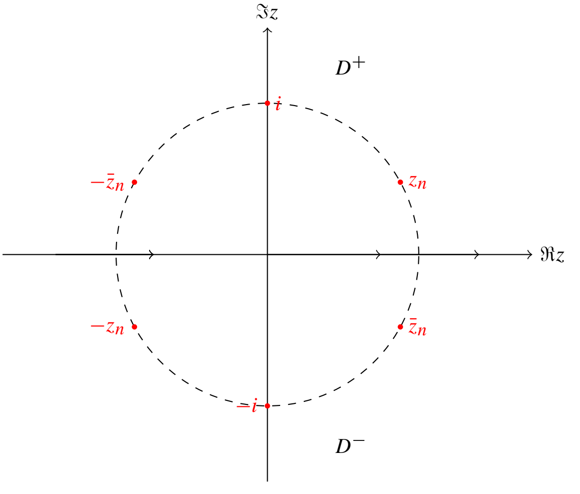


Fig. 2. The discrete spectrums distribute on the unite circle $\{z : |z| = 1\}$ on the z-plane.

which implies the result. $\square$

In a similar way [13], we can show that zeros of $a(z)$ are finite and simple, all of which are placed on the unit circle $\{z : |z| = 1\}$ (see Fig. 2). Suppose that $a(z)$ has finite N simple zeros $z_1, z_2, \dots, z_N$ on $D_+ \cap \{z : |z| = 1, \Im z > 0, \Re z > 0\}$.

The symmetries of $S(z)$ imply that

$$a(z_n) = 0 \Leftrightarrow a(\bar{z}_n) = 0 \Leftrightarrow a(-z_n) = 0 \Leftrightarrow a(-\bar{z}_n) = 0, \quad n = 1, \dots, N.$$

Therefore we give the discrete spectrum by

$$\mathcal{Z} = \{z_n, \bar{z}_n - \bar{z}_n, -z_n\}_{n=1}^N, \tag{2.33}$$

where z_n satisfies that $|z_n| = 1$, $\Re z_n > 0$, $\Im z_n > 0$. Moreover, it is convenient to define that

$$\eta_n = \begin{cases} z_n, & n = 1, \dots, N, \\ -\bar{z}_{n-N}, & n = N+1, \dots, 2N, \end{cases} \tag{2.34}$$

from which we express the set $\mathcal{Z}$ in terms of

$$\mathcal{Z} = \{\eta_n, \bar{\eta}_n\}_{n=1}^{2N}. \tag{2.35}$$

Using trace formulae, $a(z)$ is given by

$$a(z) = \prod_{n=1}^{2N} \left(\frac{z - \eta_n}{z - \bar{\eta}_n}\right) \exp\left[-\frac{1}{2\pi i} \int_\Sigma \frac{\log(1 - r(s)\tilde{r}(s))}{s - z} ds\right], \quad z \in \mathbb{C}_+. \tag{2.36}$$

Denoting norming constant $c_n = b_n/a'(\eta_n)$, the residue condition follows immediately

$$\operatorname*{Res}_{z=\eta_n}\left[\frac{\mu_{+,1}(z)}{a(z)}\right] = c_n e^{-2i\lambda(\eta_n)x}\mu_{-,2}(\eta_n), \tag{2.37}$$

$$\operatorname*{Res}_{z=\bar{\eta}_n}\left[\frac{\mu_{+,2}(z)}{\overline{a(\bar{z})}}\right] = \bar{c}_n e^{2i\lambda(\bar{\eta}_n)x}\mu_{-,1}(\bar{\eta}_n). \tag{2.38}$$

Collect $\sigma_d = \{\eta_n, c_n\}_{n=1}^{2N}$ the scattering data. Now we try to carry out the time evolution of the scattering data. If q also depends on time variable t, we can obtain the functions $a(z)$ and $b(z)$ mentioned above for all times $t \in \mathbb{R}$. Applying ∂_t to (2.1) and taking some standard arguments, such as [15,16], we know that time dependence of scattering data could be expressed in terms of the following replacement

$$c(\eta_n) \to c(t, \eta_n) = c(0, \eta_n)e^{\lambda(\eta_n)(4k^2(\eta_n)+2)t}, \tag{2.39}$$

$$r(z) \to r(t, z) = r(0, z)e^{\lambda(4k^2+2)t}. \tag{2.40}$$

Remark 2.8. At time $t = 0$, the initial function $q(x, 0)$ produces $4N$ simple zeros of $a(z, 0)$. If q evolves in terms of (1.1), then $q(x, t)$ will produce exactly the same $4N$ simple zeros at time $0 \neq t \in \mathbb{R}$ for $a(z, t)$. And the scattering data with time variable t can be given by

$$\left\{r(z)e^{\lambda(4k^2+2)t}, \{\eta_n, c(\eta_n)e^{\lambda(\eta_n)(4k^2(\eta_n)+2)t}\}_{n=1}^{2N}\right\},$$

where $\left\{r(z), \{\eta_n, c_n\}_{n=1}^{2N}\right\}$ are corresponded to initial data $q_0(x)$.

2.2. Set up of the Riemann-Hilbert problem

Define a sectionally meromorphic matrix as follows

$$M(x, t; z) := \begin{cases} \left(\dfrac{\mu_{+,1}(x, t; z)}{a(z)}, \mu_{-,2}(x, t; z)\right), & z \in \mathbb{C}_+ \\ \left(\mu_{-,1}(x, t; z), \dfrac{\mu_{+,2}(x, t; z)}{\overline{a(\bar{z})}}\right), & z \in \mathbb{C}_-, \end{cases} \tag{2.41}$$

which solves the following RH problem.

RH problem 2.9. *Find a 2×2 matrix-valued function $M(x, t; z)$ such that*
- $M(z)$ is analytical in $\mathbb{C}\backslash(\Sigma \cup \mathcal{Z})$ and has simple poles in $\mathcal{Z} = \{\eta_n, \bar{\eta}_n\}_{n=1}^{2N}$.
- $M(z) = \sigma_1\overline{M(\bar{z})}\sigma_1 = \overline{M(-\bar{z})} = \mp z^{-1}M(z^{-1})\sigma_2$.
- The non-tangential limits $M_\pm(z) = \lim_{s\to z} M(s)$, $s \in \mathbb{C}_\pm$ exist for any $z \in \Sigma$ and satisfy the jump relation $M_+(z) = M_-(z)V(z)$ where

$$V(z) = \begin{pmatrix} 1 - |r(z)|^2 & -\overline{r(z)}e^{2it\theta} \\ r(z)e^{-2it\theta} & 1 \end{pmatrix}, \quad z \in \Sigma, \tag{2.42}$$

with $\theta(z) = \lambda(z)\left[\frac{x}{t} + 4k^2(z) + 2\right]$.
- Asymptotic behavior

$$M(x,t;z) = I + \mathcal{O}(z^{-1}), \quad z \to \infty, \tag{2.43}$$

$$M(x,t;z) = \frac{\sigma_2}{z} + \mathcal{O}(1), \quad z \to 0. \tag{2.44}$$

- Residue conditions

$$\operatorname*{Res}_{z=\eta_n} M(z) = \lim_{z\to\eta_n} M(z) \begin{pmatrix} 0 & 0 \\ c_n e^{-2it\theta(\eta_n)} & 0 \end{pmatrix}, \tag{2.45}$$

$$\operatorname*{Res}_{z=\bar{\eta}_n} M(z) = \lim_{z\to\bar{\eta}_n} M(z) \begin{pmatrix} 0 & \bar{c}_n e^{2it\theta(\bar{\eta}_n)} \\ 0 & 0 \end{pmatrix}. \tag{2.46}$$

The solution $q(x,t)$ of (1.1) can be expressed in terms of the solution M of RH problem 2.9 via the following proposition.

Proposition 2.10. *Assuming* $q - \tanh(x) \in L^{1,2}(\mathbb{R})$ *and* $q'(x) \in W^{1,1}(\mathbb{R})$, *we have the following asymptotics of* $M(z)$ *as* $z \to \infty$ *and* $z \to 0$*:*

$$\lim_{z\to\infty} z(M(z) - I) = \begin{pmatrix} -i\int_x^\infty (q^2-1)dx & iq \\ -iq & i\int_x^\infty (q^2-1)dx \end{pmatrix}, \tag{2.47}$$

$$\lim_{z\to 0}(M(z) - \frac{\sigma_2}{z}) = \begin{pmatrix} iq & -i\int_x^\infty (q^2-1)dx \\ i\int_x^\infty (q^2-1)dx & -iq \end{pmatrix}. \tag{2.48}$$

And the solution $q(x,t)$ *of* (1.1)-(1.2) *is given by*

$$q(x,t) = -i(M_1)_{12} = -i\lim_{z\to\infty}(zM)_{12}, \tag{2.49}$$

where M_1 *appears in the expansion of* $M = I + z^{-1}M_1 + \mathcal{O}(z^{-2})$ *as* $z \to \infty$.

Proof. This proposition follows from the third item of Proposition 2.2. □

3. Distribution of saddle points and signature table

The exponential term appeared in the jump matrix of RH problem 2.9 plays a key role in our analysis.

$$e^{\pm 2it\theta}, \quad \theta(z) = \frac{1}{2}\left(z - \frac{1}{z}\right)\left[\frac{x}{t} + 2 + \left(z + \frac{1}{z}\right)^2\right]. \tag{3.1}$$

In this section, we present the analysis on the phase function $\theta(z)$, which include the saddle points (see Fig. 3) and the signature tables for $e^{2it\theta(z)}$ (see Fig. 4). Direct calculation shows that:

$$\Im\theta(z) = \frac{\xi+3}{2}\Im z - \frac{\xi+3}{2|z|^2}\Im\bar{z} + \frac{1}{2}\Im(z^3) - \frac{1}{2|z|^6}\Im(\bar{z}^3), \tag{3.2}$$

$$\Re(2it\theta(z)) = -t\left[(\xi+3)\Im z - (\xi+3)|z|^{-2}\Im\bar{z} + \Im(z^3) - |z|^{-6}\Im(\bar{z}^3)\right]. \tag{3.3}$$

To find the stationary phase points (or saddle points), we need $\theta'(z)$

$$\theta'(z) = \frac{3}{2}z^2 + \frac{\xi+3}{2z^2} + \frac{3}{2z^4} + \frac{\xi+3}{2}. \tag{3.4}$$

Proposition 3.1 *(Distribution of saddle points). Besides two fixed saddle points $i, -i$, there exist four saddle points which satisfy the following properties for different ξ (see Fig. 3):*
- For $\xi < -6$, the four saddle points $\xi_j := \xi_j(\xi),\ j = 1,2,3,4$ are located on the jump contour $\Sigma = \mathbb{R}\backslash\{0\}$. Moreover, we have $\xi_4 < -1 < \xi_3 < 0 < \xi_2 < 1 < \xi_1$ and $\xi_1 = \frac{1}{\xi_2} = -\frac{1}{\xi_3} = -\xi_4$;
- For $-6 < \xi < 6$, the four saddle points are away from the coordinate axis (both real and imaginary axis);
- For $\xi > 6$, the four saddle points are all located on the imaginary axis. Moreover, we have $\Im\xi_1 > 1 > \Im\xi_2 > 0 > \Im\xi_3 > -1 > \Im\xi_4$ and $\xi_1\xi_2 = \xi_3\xi_4 = -1$.

Proof. From $\theta'(z) = 0$, we have

$$3z^6 + (\xi+3)z^4 + (\xi+3)z^2 + 3 = 0. \tag{3.5}$$

Using factorization technique, we obtain

$$(1+z^2)\left(3z^4 + \xi z^2 + 3\right) = 0. \tag{3.6}$$

From the equality, we have two fixed saddle points $i, -i$. And we can solve that

$$z^2 = -\frac{\xi + \sqrt{\xi^2-36}}{6}, \quad \text{or} \quad z^2 = -\frac{\xi - \sqrt{\xi^2-36}}{6}. \tag{3.7}$$

For $\xi < -6$, both $-\frac{\xi+\sqrt{\xi^2-36}}{6}$ and $-\frac{\xi-\sqrt{\xi^2-36}}{6}$ are greater than zero, and four roots are as follows.

$$\xi_1 = \sqrt{-\frac{\xi - \sqrt{\xi^2-36}}{6}}, \quad \xi_4 = -\sqrt{-\frac{\xi - \sqrt{\xi^2-36}}{6}}, \tag{3.8}$$

$$\xi_2 = \sqrt{-\frac{\xi + \sqrt{\xi^2-36}}{6}}, \quad \xi_3 = -\sqrt{-\frac{\xi + \sqrt{\xi^2-36}}{6}}, \tag{3.9}$$

with relation $\xi_4 < -1 < \xi_3 < 0 < \xi_2 < 1 < \xi_1$ and $\xi_1 = \frac{1}{\xi_2} = -\frac{1}{\xi_3} = -\xi_4$.

For $-6 < \xi < 6$, the discriminant $\xi^2 - 36$ is less than zero. We can know that there exist four saddle points $\xi_j = \Re(\xi_j) + i\Im(\xi_j)$, where $\Re(\xi_j), \Im(\xi_j) \neq 0,\ j = 1,2,3,4$.

For $\xi > 6$, both $-\frac{\xi+\sqrt{\xi^2-36}}{6}$ and $-\frac{\xi-\sqrt{\xi^2-36}}{6}$ are less than zero. And four pure imaginary saddle points are as follows

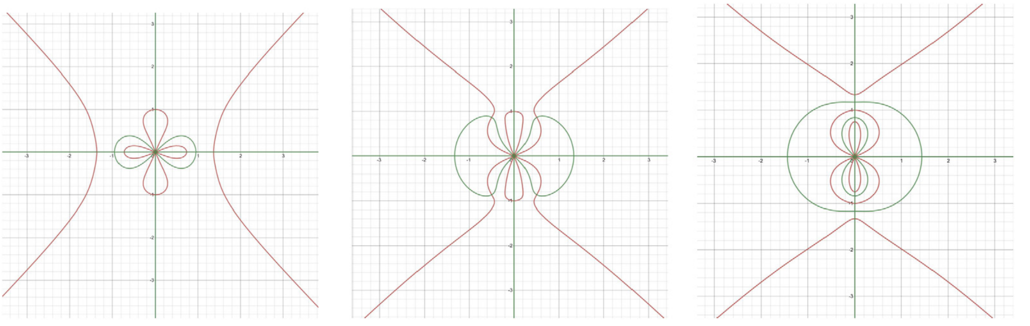


Fig. 3. Plots of the distributions for saddle points: **(a)** $\xi < -6$, **(b)** $-6 < \xi < 6$, **(c)** $\xi > 6$. The red curve represents $\Re\theta'(z) = 0$, and the green curve represents $\Im\theta'(z) = 0$. The intersection points are the saddle points which represent the zeros of $\theta'(z) = 0$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

$$\xi_1 = i\sqrt{\frac{\xi + \sqrt{\xi^2 - 36}}{6}}, \quad \xi_4 = -i\sqrt{\frac{\xi + \sqrt{\xi^2 - 36}}{6}}, \tag{3.10}$$

$$\xi_2 = i\sqrt{\frac{\xi - \sqrt{\xi^2 - 36}}{6}}, \quad \xi_3 = -i\sqrt{\frac{\xi - \sqrt{\xi^2 - 36}}{6}}, \tag{3.11}$$

with $\Im\xi_1 > 1 > \Im\xi_2 > 0 > \Im\xi_3 > -1 > \Im\xi_4$ and $\xi_1\xi_2 = \xi_3\xi_4 = -1$. $\square$

Remark 3.2. We can see that, for example, ξ_1, ξ_4 defined by (3.8) can't be ∞ because of the finite $\xi = \mathcal{O}(1)$.

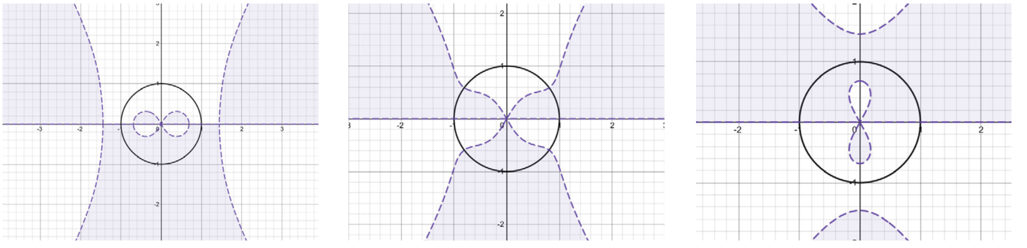
According to the Fig. 3 and Fig. 4. We can find that: for $\xi < -6$, there exist four stationary phase points besides $i, -i$, which are all located on the jump contour Σ as shown in Fig. 3(a) with signature table shown in Fig. 4(a). For $-6 < \xi < -2$, the distribution of phase points is shown in Fig. 3(b) and signature table is shown in Fig. 4(b). For $\xi > -2$, there exist four stationary phase points besides $i, -i$. When $-2 < \xi < 6$, the four saddle points are away from the coordinate axis (both real and imaginary axis), which is corresponded to Fig. 3(b) and the signature table is shown in Fig. 4(c). The asymptotics for $-2 < \xi < 6$ could be seen as a specific case of the asymptotics for $-6 < \xi < -2$. For $\xi > 6$, the four saddle points are all distributed on the imaginary axis as shown in Fig. 3(c) and the signature table is still shown in Fig. 4(c).

4. Asymptotics for $\xi \in \mathcal{R}_L$: left field

4.1. Jump matrix factorizations

Now we use factorizations of the jump matrix along the real axis to deform the contours onto those on which the oscillatory jump on the real axis is traded for exponential decay. This step is aided by two well known factorizations of the jump matrix $V(z)$ in (2.42):

$$V(z) = \begin{pmatrix} 1 & -\overline{r(z)}e^{2it\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r(z)e^{-2it\theta} & 1 \end{pmatrix}, \quad z \in \tilde{\Gamma}, \tag{4.1}$$



(a) (b) (c)

Fig. 4. Plots of the $\Im\theta$ with different $\xi = x/t$: **(a)** $\xi < -6$, **(b)** $-6 < \xi < -2$, **(c)** $\xi > 6$. The black curve is unit circle. In the purple region, $\Im\theta > 0$ ($|e^{2it\theta}| \to 0$ as $t \to \infty$), and $\Im\theta < 0$ ($|e^{-2it\theta}| \to 0$ as $t \to \infty$) in the white region. The purple dotted curve represents $\Im\theta = 0$.

$$= \begin{pmatrix} 1 & 0 \\ \frac{r(z)e^{-2it\theta}}{1-|r(z)|^2} & 1 \end{pmatrix} \left(1 - |r(z)|^2\right)^{\sigma_3} \begin{pmatrix} 1 & -\frac{\overline{r(z)}e^{2it\theta}}{1-|r(z)|^2} \\ 0 & 1 \end{pmatrix}, \quad z \in \Gamma, \tag{4.2}$$

where

$$\Gamma := (-\infty, \xi_4) \cup (\xi_3, 0) \cup (0, \xi_2) \cup (\xi_1, +\infty),$$
$$\tilde{\Gamma} := (\xi_4, \xi_3) \cup (\xi_2, \xi_1).$$

The leftmost term of the factorization can be deformed into $\mathbb{C}_-$, the rightmost term can be deformed into $\mathbb{C}_+$, while any central terms remain on the real axis. These deformations are useful when they deformed the factors into regions in which the corresponding off-diagonal exponential terms $e^{\pm 2it\theta}$ are decaying as $t \to \infty$.

4.2. *First transformation: $M \to M^{(1)}$*

Define the function

$$\delta(z) := \delta(z, \xi) = \exp\left[-\frac{1}{2\pi i}\int_\Gamma \log\left(1 - |r(s)|^2\right)\left(\frac{1}{s-z} - \frac{1}{2s}\right) ds\right]. \tag{4.3}$$

Taking $\nu(z) = -\frac{1}{2\pi}\log(1 - |r(z)|^2)$, then we can express

$$\delta(z) = \exp\left(-i\int_\Gamma \nu(s)\left(\frac{1}{s-z} - \frac{1}{2s}\right) ds\right). \tag{4.4}$$

In the above formulaes, we choose the principal branch of power and logarithm functions.

Proposition 4.1. *The function defined by* (4.4) *admits following properties:*

(i) $\delta(z)$ *is analytical for* $z \in \mathbb{C}\setminus\Gamma$*;*
(ii) $\delta_-(z, \xi) = \delta_+(z, \xi)\left(1 - |r(z)|^2\right)$, $z \in \Gamma$*;*

(iii). $\delta(z) = \overline{\delta^{-1}(\bar{z})} = \overline{\delta(-\bar{z})} = \delta(z^{-1})^{-1}$;
(iv). $\delta(\infty) := \lim_{z\to\infty}\delta(z) = 1$. *And* $\delta(z)$ *is continuous at* $z = 0$ *with* $\delta(0) = \delta(\infty) = 1$;
(v) $\delta(z)$ *is uniformly bounded in* $\mathbb{C} \setminus \mathbb{R}$

$$(1-\rho^2)^{1/2} \leqslant |\delta(z)| \leqslant (1-\rho^2)^{-1/2}, \tag{4.5}$$

where $||r||_{L^\infty} \leqslant \rho < 1$.
(vi) *As* $z \to \xi_j$ *along any ray* $\xi_j + e^{i\phi}\mathbb{R}_+$ *with* $|\phi| < \pi$, *we have*

$$|\delta(z) - (z-\xi_j)^{i\epsilon_j \nu(\xi_j)} e^{i\beta(z,\xi_j)}| \leqslant c\|r\|_{H^1}|z-\xi_j|^{\frac{1}{2}}, \tag{4.6}$$

$$||\beta||_{L^\infty} \leqslant \frac{c||r||_{H^{1,0}}}{1-\rho}, \tag{4.7}$$

$$|\beta(z,\xi) - \beta(\xi_j,\xi)| < \frac{c||r||_{H^{1,0}}}{1-\rho}|z-\xi_j|^{\frac{1}{2}}, \tag{4.8}$$

where

$$\beta(z,\xi_j) = \int_\Gamma \frac{\nu(s)}{s-z}ds + \epsilon_j \nu(\xi_j)\log(z-\xi_j). \tag{4.9}$$

Proof. Properties of (i), (iv) can be obtained by simple calculation from the definition of $\delta(z)$. The jump relation (ii) follows from the Plemelj formulae. As for the property (iii), the symmetry comes from the symmetry of $r(z)$. We specially point out that the third symmetry follows from the symmetry of $r(z)$ as well as the following equality

$$\exp\left(i\int_\Gamma \frac{\nu(s)}{s-z}ds\right) = \exp\left[i\int_\Gamma \nu(s)\left(\frac{1}{s-z} - \frac{1}{2s}\right)ds\right], \quad \text{for} \quad z \in \Gamma. \tag{4.10}$$

For the item (v) and item (vi), the analysis is similar to [12, Lemma 3.1]. □

Furthermore, we can rewrite

$$\delta(z,\xi) = (z-\xi_j)^{i\nu(\xi_j)}\exp(i\beta(z,\xi)). \tag{4.11}$$

Remark 4.2. We notice that all discrete spectrums $\eta_n \in \mathbb{C}_+ \cap \{z : |z| = 1\}$ satisfy $\Im\theta(\eta_n) < 0$, all discrete spectrums $\bar{\eta}_n \in \mathbb{C}_- \cap \{z : |z| = 1\}$ satisfy $\Im\theta(\bar{\eta}_n) > 0$. Owe to this good property, that's why we do not classify the discrete spectrum by $\delta(z,\xi)$, which is different from the $T(z)$ we use in [35].

Introduce the interpolation functions which can convert the residue conditions (2.45) and (2.46) into the jump condition. For all poles $\eta_j \in \mathcal{Z}$, we define a constant h as follows

$$h := \frac{1}{2}\min\left\{\min_{i\neq j}|\eta_i - \eta_j|,\ \min_{j\in\mathcal{Z}}|\Im\eta_j|\right\}. \tag{4.12}$$

We can see that the disk $D(\eta_i, h) \cap D(\eta_j, h) = \emptyset$ for $i \neq j$, and $D(\eta_i, h) \cap \mathbb{R} = \emptyset$. To be brief, we define a new path

$$\Sigma^{pole} = \bigcup_{n=1}^{N} \{z \in \mathbb{C} : z \in \partial D(\eta_n, h) \text{ or } z \in \partial D(\bar{\eta}_n, h)\}. \tag{4.13}$$

The interpolation function $G(z)$ is introduced by

$$G(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ -\frac{c_n e^{-2it\theta(\eta_n)}}{z-\eta_n} & 1 \end{pmatrix}, & z \in D(\eta_n, h), \\ \begin{pmatrix} 1 & -\frac{\bar{c}_n e^{2it\theta(\bar{\eta}_n)}}{z-\bar{\eta}_n} \\ 0 & 1 \end{pmatrix}, & z \in D(\bar{\eta}_n, h), \\ I & elsewhere. \end{cases} \tag{4.14}$$

By using $\delta(z, \xi)$ and the interpolation function $G(z)$, the new matrix-valued function $M^{(1)}(z)$ is defined by

$$M^{(1)}(x, t; z) := M^{(1)}(z) = M(z)G(z)\delta(z)^{\sigma_3}, \tag{4.15}$$

which satisfies the following regular RH problem.

RH problem 4.3. *Find a 2×2 matrix-valued function $M^{(1)}(x, t; z)$ such that*
- *$M^{(1)}(z)$ is analytic for $\mathbb{C} \setminus \Sigma^{(1)}$, where $\Sigma^{(1)} = \Sigma \cup \Sigma^{pole}$.*
- *$M^{(1)}(z) = \sigma_1 \overline{M^{(1)}(\bar{z})}\sigma_1 = \overline{M^{(1)}(-\bar{z})} = \mp z^{-1} M^{(1)}(z^{-1})\sigma_2$.*
- *The non-tangential limits $M^{(1)}_{\pm}(z)$ exist for any $z \in \Sigma^{(1)}$ and satisfy the jump relation $M^{(1)}_{+}(z) = M^{(1)}_{-}(z)V^{(1)}(z)$ where*

$$V^{(1)}(z) = \begin{cases} \begin{pmatrix} 1 & -\overline{r(z)}\delta(z)^{-2}e^{2it\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r(z)\delta^2(z)e^{-2it\theta} & 1 \end{pmatrix}, & z \in \tilde{\Gamma}, \\ \begin{pmatrix} 1 & 0 \\ \frac{r(z)\delta_-^2(z)}{1-|r(z)|^2}e^{-2it\theta} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{\overline{r(z)}\delta_+^{-2}(z)}{1-|r(z)|^2}e^{2it\theta} \\ 0 & 1 \end{pmatrix}, & z \in \Gamma, \\ \begin{pmatrix} 1 & 0 \\ -\frac{c_n e^{-2it\theta(\eta_n)}\delta^{-2}}{z-\eta_n} & 1 \end{pmatrix}, & z \in \partial D(\eta_n, h) \text{ oriented counterclockwise}, \\ \begin{pmatrix} 1 & \frac{\bar{c}_n e^{2it\theta(\bar{\eta}_n)}\delta^2(z)}{z-\bar{\eta}_n} \\ 0 & 1 \end{pmatrix}, & z \in \partial D(\bar{\eta}_n, h) \text{ oriented clockwise}. \end{cases} \tag{4.16}$$

- *Asymptotic behavior*

$$M^{(1)}(x, t; z) = I + \mathcal{O}(z^{-1}), \quad z \to \infty, \tag{4.17}$$

$$M^{(1)}(x, t; z) = \frac{\sigma_2}{z} + \mathcal{O}(1), \quad z \to 0. \tag{4.18}$$

4.3. Second transformation by opening $\bar{\partial}$ lenses: $M^{(1)} \to M^{(2)}$

In this subsection, we make continuous extension for the jump matrix $V^{(1)}(z)$ to remove the jump from the real axis in such a way that the new problem takes advantage of the decay of $\exp(\pm 2it\theta)$ for $z \notin \mathbb{R}$.

4.3.1. Characteristic lines

The aim of this subsubsection is to denote some characteristic lines which are the jump contours of the RH problem $M^{(2)}$ defined below. To avoid these characteristic lines intersect with discrete spectrum located on the unit circle, we fix a small enough angle θ_0 which satisfies that the following three conditions.
- Let the set

$$\left\{ z \in \mathbb{C} : \tan\theta_0 < \left| \frac{\Im z}{\Re z - \xi_j} \right| \right\} \tag{4.19}$$

do not intersect the set $\mathcal{Z}$, $j = 2, 3$ for $\xi \in \mathcal{R}_L$;
- The following regions Ω_{jk}, $j = 2, 3, k = 3, 4$ do not intersect discrete spectrums, which implies that

$$\Upsilon(\xi) = \min\left\{ \theta_0, \frac{\pi}{4} \right\}, \tag{4.20}$$

- Recalling Proposition 3.1, we make

$$d \in \left(0, \frac{\xi_2}{2\cos\Upsilon}\right), \quad \tilde{d} \in \left(0, \frac{\xi_1 - \xi_2}{2\cos\Upsilon}\right). \tag{4.21}$$

With these conditions, some characteristic lines are given as follows items (For convenience, see Fig. 6).

(i) For an angle ϕ satisfies the above conditions (4.19), (4.20) and (4.21), we denote the characteristic lines near saddle points presented by

$$\Sigma_{j1} = \begin{cases} \xi_j + e^{i\left[(j-1)\pi + (-1)^{j-1}\phi\right]}\mathbb{R}_+, & j = 1, 4, \\ \xi_j + e^{i\left[(j-1)\pi + (-1)^{j-1}\phi\right]}d, & j = 2, 3, \end{cases}$$

$$\Sigma_{j2} = \begin{cases} \xi_j + e^{-i\left[(j-1)\pi + (-1)^{j-1}\phi\right]}\mathbb{R}_+, & j = 1, 4, \\ \xi_j + e^{-i\left[(j-1)\pi + (-1)^{j-1}\phi\right]}d, & j = 2, 3, \end{cases} \tag{4.22}$$

$$\Sigma_{j3} = \begin{cases} \xi_j + e^{-i\left[j\pi + (-1)^{j}\phi\right]}\tilde{d}, & j = 1, 4, \\ \xi_j + e^{-i\left[j\pi + (-1)^{j}\phi\right]}\tilde{d}, & j = 2, 3, \end{cases}$$

$$\Sigma_{j4} = \begin{cases} \xi_j + e^{i\left[j\pi + (-1)^{j}\phi\right]}\tilde{d}, & j = 1, 4, \\ \xi_j + e^{i\left[j\pi + (-1)^{j}\phi\right]}\tilde{d}, & j = 2, 3. \end{cases} \tag{4.23}$$

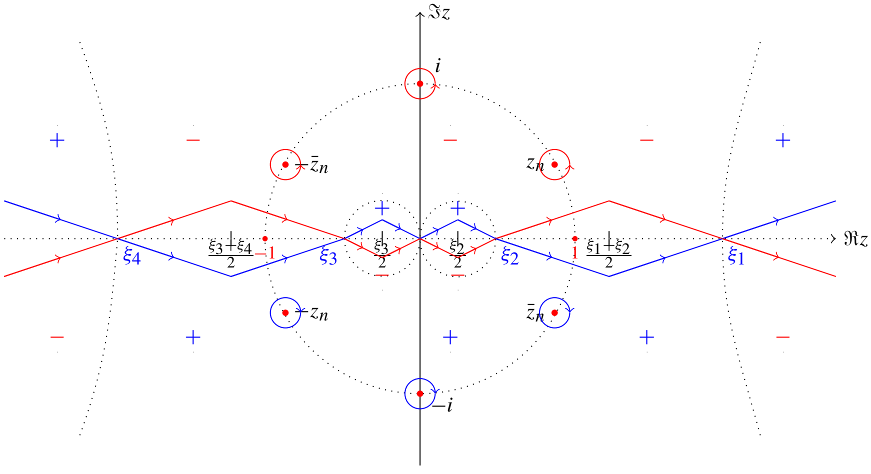


Fig. 5. There are four stationary phase points $\xi_1, \ldots, \xi_4$ with $\xi_1 = -\xi_4 = 1/\xi_2 = -1/\xi_3$ for $\xi \in \mathcal{R}_L$. Open jump contour $\mathbb{R}\backslash\{0\}$ such that red and blue lines don't intersect the discrete spectrum on the unite circle $|z| = 1$. Additionally, the blue "+" implies that $e^{2it\theta} \to 0$ as $t \to +\infty$, on the other side, the red "−" implies that $e^{-2it\theta} \to 0$ as $t \to +\infty$.

(ii) Characteristic lines near $z = 0$ are defined as follows

$$\Sigma_{j1} = \begin{cases} e^{i\phi}d, & j = 0^+, \\ e^{i(\pi-\phi)}d, & j = 0^-, \end{cases} \qquad \Sigma_{j2} = \begin{cases} e^{-i\phi}d, & j = 0^+, \\ e^{-i(\pi-\phi)}d, & j = 0^-. \end{cases} \tag{4.24}$$

(iii) Meanwhile, there exist vertical jumps $\Sigma_{j\pm}^{(1/2)}$, $j = 1, 2, 3, 4$.

The complex plane $\mathbb{C}$ is separated by these contours, which is shown in Fig. 5, Fig. 6. Denote

$$\Omega = \left(\bigcup_{j,k=1,2,3,4} \Omega_{jk} \right) \bigcup \left(\bigcup_{k=1,2} \Omega_{0\pm k} \right), \quad \Omega_{\pm} = \mathbb{C}\backslash\Omega, \tag{4.25}$$

$$\Sigma^{(2)} := \left(\bigcup_{j,k=1,2,3,4} \Sigma_{jk} \right) \bigcup \left(\bigcup_{k=1,2} \Sigma_{0\pm k} \right) \bigcup \left(\bigcup_{j=1,2,3,4} \Sigma_{j\pm}^{(1/2)} \right) \bigcup \Sigma^{pole}. \tag{4.26}$$

4.3.2. Some estimations for $\Im\theta(z)$

In this subsubsection, we give some estimations for $\Im\theta(z)$ in different regions.

Proposition 4.4 *(near $z = 0$). For a fixed small angle ϕ which satisfies (4.19), (4.20) and (4.21), the imaginary part of phase function $\theta(z)$ defined by (3.1) has following estimations:*

$$\Im\theta(z) \geqslant c|\sin\phi|\sqrt{\alpha}, \quad \text{as} \quad z \in \Omega_{0\pm1}, \tag{4.27}$$

$$\Im\theta(z) \leqslant -c|\sin\phi|\sqrt{\alpha}, \quad \text{as} \quad z \in \Omega_{0\pm2}, \tag{4.28}$$

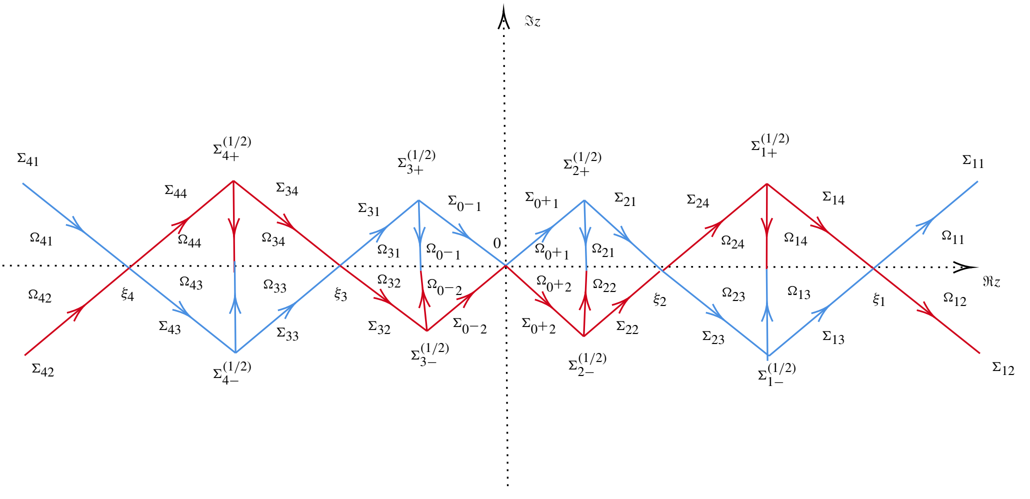


Fig. 6. Σ_{jk} separate complex $\mathbb{C}$ into some regions denoted by Ω_{jk}.

where

$$c = c(\xi) > 0, \quad \alpha = 3 - \frac{\xi + 3}{1 + 2\cos(2\phi)}. \tag{4.29}$$

Proof. We present the details for $z \in \Omega_{0+1}$, the others are similar. Taking $z = le^{i\phi}$, we can rewrite the (3.2) as

$$\Im\theta(z) = \frac{1}{2} F(l)\sin\phi \left[\xi - 6\cos(2\phi) + (2\cos(2\phi) + 1)F^2(l)\right], \tag{4.30}$$

where $F(l) = l + l^{-1} \geqslant 2$. Firstly we calculate the critical situation $\Im\theta(z) = 0$. Taking (4.30), $F(l) \geqslant 2$ as well as $\sin\phi > 0$, we have

$$\xi - 6\cos(2\phi) + [2\cos(2\phi) + 1]\, F^2(l) = 0. \tag{4.31}$$

Thus

$$F^2(l) = 3 - \frac{\xi + 3}{1 + 2\cos(2\phi)} =: \alpha > 4. \tag{4.32}$$

Moreover, by $F(l) = \sqrt{\alpha}$, we have $l^2 - \sqrt{\alpha} l + 1 = 0$. Solving this quadratic equation, we obtain two roots

$$l_1 = \frac{\sqrt{\alpha} - \sqrt{\alpha - 4}}{2} < l_2 = \frac{\sqrt{\alpha} + \sqrt{\alpha - 4}}{2}. \tag{4.33}$$

We claim that: $\Im\theta(z) > 0$ as $l < l_1$ (corresponding to $z \in \Omega_{0+1}$). It's easy to check that $h(l) := l^2 - \sqrt{\alpha} l + 1$ is monotonically increasing on the $(\sqrt{\alpha}/2, +\infty)$, while monotonically decreasing on the $(-\infty, \sqrt{\alpha}/2)$. Since $l_1 < \sqrt{\alpha}/2$, $h(l)$ is monotonically decreasing on the $(0, l_1)$. Thus we have $h(l) > h(l_1) = 0$ and $F(l) > \sqrt{\alpha}$, which implies that

$$\Im\theta(z) > \frac{1}{2}\sqrt{\alpha}\sin\phi\,[\xi - 6\cos(2\phi) + (2\cos(2\phi)+1)\alpha] = 0. \tag{4.34}$$

Thus we bring this proof to an end. $\square$

Corollary 4.5. *$\Im\theta(z)$ has following evaluation for $z = le^{i\phi} := u_0 + iv$*

$$\Im\theta(z) \geqslant cv, \quad \text{for} \quad z \in \Omega_{0\pm 1}, \tag{4.35}$$

$$\Im\theta(z) \leqslant -cv, \quad \text{for} \quad z \in \Omega_{0\pm 2}, \tag{4.36}$$

where $c = c(\xi) > 0$.

Proposition 4.6 *(Near $z = \xi_j$, $j = 2, 3$). For a fixed small angle ϕ (the same in Proposition 4.4) which satisfies (4.19), (4.20) and (4.21), the imaginary part of phase function $\theta(z)$ defined by (3.1) has following estimations:*

$$\Im\theta(z) \leqslant -c\left(1 + |z|^{-2}\right)v^2, \quad z \in \Omega_{jk}, \quad j = 2, 3, \quad k = 2, 4 \tag{4.37}$$

$$\Im\theta(z) \geqslant c\left(1 + |z|^{-2}\right)v^2, \quad z \in \Omega_{jk}, \quad j = 2, 3, \quad k = 1, 3, \tag{4.38}$$

where $c = c(\xi) > 0$.

Proof. Taking $z \in \Omega_{24}$ as an example, the proof for the other regions is similar. Denoting $z = \xi_2 + le^{i\phi} := \xi_2 + u_2 + iv$, we can rewrite (3.2) as

$$\begin{aligned}\Im\theta(z) &= \frac{v}{2}\left(1 + |z|^{-2}\right)\left[\xi + 3 + 3(|z|^2 + |z|^{-2} - 1) - 4v^2(1 + |z|^{-4} - |z|^{-2})\right] \\ &\leqslant c\left(1 + |z|^{-2}\right)\left[\xi + 3 + 3(|z|^2 + |z|^{-2} - 1) - 4v^2(1 + |z|^{-4} - |z|^{-2})\right],\end{aligned} \tag{4.39}$$

the second step we used $v \leqslant \frac{\xi_1 - \xi_2}{2}$.

Consider

$$h\left(|z|^2\right) := \left[\xi + 3 + 3(|z|^2 + |z|^{-2} - 1) - 4v^2(1 + |z|^{-4} - |z|^{-2})\right]. \tag{4.40}$$

Taking $\tau = |z|^2 \in \left(\xi_2^2, \xi_1^2\right)$, we obtain that

$$h(\tau) = 3(\tau + \tau^{-1} - 1) - 4v^2(1 + \tau^{-2} - \tau^{-1}) + \xi + 3. \tag{4.41}$$

It is not difficult to verify that $h'(\tau) < 0$ for $\tau \in (\xi_2^2, \xi_1^2)$, thus

$$\begin{aligned} h(\tau) &\leqslant h(\xi_2^2) \\ &= 3\left(\xi_2^2+\xi_2^{-2}-1\right)-4v^2\left(1+\xi_2^{-4}-\xi_2^{-2}\right)+\xi+3 \\ &\overset{\xi_2=1/\xi_1}{=} 3\left(\xi_2^2+\xi_1^2-1\right)-4v^2\left(1+\xi_1^4-\xi_1^2\right)+\xi+3. \end{aligned} \tag{4.42}$$

Since ξ_1 is the saddle point, we have $\theta'(\xi_1)=0$. Using $\xi_1\xi_2=1$ again, we can obtain the following relation from $\theta'(\xi_1)=0$ such that

$$\xi+3=\frac{-3\left(\xi_1^2+\xi_2^4\right)}{1+\xi_2^2}. \tag{4.43}$$

With (4.43), we are lucky enough to find that

$$3\left(\xi_2^2+\xi_1^2-1\right)+\xi+3=0. \tag{4.44}$$

Then we obtain

$$h(\tau)\leqslant -4v^2\left(1+\xi_1^4-\xi_1^2\right). \tag{4.45}$$

As a consequence,

$$\Im\theta(z)\leqslant -cv^2\left(1+|z|^{-2}\right)\left(1+\xi_1^4-\xi_1^2\right)\leqslant -c(\xi)v^2\left(1+|z|^{-2}\right)<0. \quad \square \tag{4.46}$$

Proposition 4.7 *(Near $z=\xi_j$, $j=1,4$). For a fixed small angle ϕ (the same in Proposition 4.4) which satisfies (4.19), (4.20) and (4.21), the imaginary part of phase function $\theta(z)$ defined by (3.1) admits following estimations:*

$$\Im\theta(z)\geqslant cv|\Re z-\xi_j|, \quad z\in\Omega_{jk}, \quad j=1,4, \quad k=1,3 \tag{4.47}$$

$$\Im\theta(z)\leqslant -cv|\Re z-\xi_j|, \quad z\in\Omega_{jk}, \quad j=1,4, \quad k=2,4, \tag{4.48}$$

where $c=c(\xi)>0$.

Proof. The proof is similar to Proposition 4.6. $\square$

4.3.3. Opening $\bar{\partial}$ lenses

Introduce the following functions: for $j=0^{\pm},1,2,3,4$

$$p_{j1}(z):=p_{j1}(z,\xi)=\frac{\overline{r(z)}}{1-|r(z)|^2}, \quad p_{j3}(z):=-\overline{r(z)}, \tag{4.49}$$

$$p_{j2}(z):=\frac{r(z)}{1-|r(z)|^2}, \quad p_{j4}(z):=-r(z). \tag{4.50}$$

Define $R^{(2)}(z):=R^{(2)}(z,\xi)$ by

$$R^{(2)}(z)=\begin{cases}\begin{pmatrix}1 & f_{j1}e^{2it\theta}\\ 0 & 1\end{pmatrix}, & z\in\Omega_{j1}, \quad j=0^{\pm},1,2,3,4\\ \begin{pmatrix}1 & 0\\ f_{j2}e^{-2it\theta} & 1\end{pmatrix} & z\in\Omega_{j2}, \quad j=0^{\pm},1,2,3,4\\ \begin{pmatrix}1 & f_{j3}e^{2it\theta}\\ 0 & 1\end{pmatrix} & z\in\Omega_{j3}, \quad j=1,2,3,4\\ \begin{pmatrix}1 & 0\\ f_{j4}e^{-2it\theta} & 1\end{pmatrix} & z\in\Omega_{j4}, \quad j=1,2,3,4\\ I, & elsewhere,\end{cases} \tag{4.51}$$

where the functions f_{jk} are given by the following two propositions.

Proposition 4.8 *(Opening lens at $z=0$). $f_{jk}:\overline{\Omega}_{jk}\to\mathbb{C}$, $j=0^{\pm},k=1,2$ are continuous on $\overline{\Omega}_{jk}$, $j=0^{\pm},k=1,2$ with boundary values:*

$$f_{0^{\pm}1}(z)=\begin{cases}p_{j1}(z)\delta_{+}^{-2}(z), & z\in\left(\frac{\xi_3}{2},0\right)\cup\left(0,\frac{\xi_2}{2}\right),\\ 0, & z\in\Sigma_{0^{\pm}1}.\end{cases} \tag{4.52}$$

$$f_{0^{\pm}2}(z)=\begin{cases}p_{j2}(z)\delta_{-}^{2}(z), & z\in\left(\frac{\xi_3}{2},0\right)\cup\left(0,\frac{\xi_2}{2}\right),\\ 0, & z\in\Sigma_{0^{\pm}2}.\end{cases} \tag{4.53}$$

f_{jk}, $j=0^{\pm},k=1,2$ have following properties:

$$|\bar\partial f_{jk}(z)|\lesssim|p_{jk}'(|z|)|+|z|^{-1/2}, \quad z\in\Omega_{jk}, \quad j=0^{\pm},k=1,2. \tag{4.54}$$

Moreover

$$|\bar\partial f_{jk}(z)|\lesssim|p_{jk}'(|z|)|+|z|^{-1}, \quad z\in\Omega_{jk}, \quad j=0^{\pm},k=1,2. \tag{4.55}$$

Proof. We give the details for $f_{0^{+}1}(z)$, which can be constructed by

$$f_{0^{+}1}(z)=p_{0^{+}1}(z)\delta_{+}^{-2}(z)\cos(\kappa_0\arg z), \quad \kappa_0=\frac{\pi}{2\phi}. \tag{4.56}$$

Denoting $z=le^{i\varphi}$, we have $\bar\partial$-derivative $\bar\partial=\frac{1}{2}e^{i\varphi}(\partial_l+il^{-1}\partial_\varphi)$. Hence

$$\bar\partial f_{0^{+}1}(z)=\frac{e^{i\varphi}}{2}\delta_{+}^{-2}(z)\left[p_{0^{+}1}'(l)\cos(\kappa_0\varphi)-\frac{i}{l}\kappa_0\sin(\kappa_0\varphi)\,p_{0^{+}1}(l)\right]. \tag{4.57}$$

Using Cauchy-Schwarz inequality, we have

$$|p_{0+1}(l)| = |p_{0+1}(l) - p_{0+1}(0)| = \left| \int_0^l p'_{0+1}(s)ds \right| \leqslant \|p'_{0+1}\|_{L^2} l^{1/2} \lesssim l^{1/2}. \tag{4.58}$$

Meanwhile, the boundedness of $\delta_+^2(z)$ is guaranteed by the property (v) of Proposition 4.1. Thus (4.54) comes true. As for (4.55), we just notice $p_{0+1}(l) \in L^\infty$. $\square$

Proposition 4.9 *(Opening lens at saddle points). $f_{jk} : \overline{\Omega}_{jk} \to \mathbb{C}$, $j, k = 1, 2, 3, 4$ are continuous on $\overline{\Omega}_{jk}$, $j, k = 1, 2, 3, 4$ with boundary values:*

$$f_{j1}(z) = \begin{cases} p_{j1}(z)\delta_+^{-2}(z), & z \in I_{j1}, \\ p_{j1}(\xi_j)e^{-2i\beta(\xi_j,\xi)}\left(z - \xi_j\right)^{-2i\epsilon_j\nu(\xi_j)}, & z \in \Sigma_{j1}, \end{cases} \tag{4.59}$$

$$f_{j2}(z) = \begin{cases} p_{j2}(z)\delta_-^{2}(z), & z \in I_{j2}, \\ p_{j2}(\xi_j)e^{2i\beta(\xi_j,\xi)}\left(z - \xi_j\right)^{2i\epsilon_j\nu(\xi_j)}, & z \in \Sigma_{j2}, \end{cases} \tag{4.60}$$

$$f_{j3}(z) = \begin{cases} p_{j3}(z)\delta^{-2}(z), & z \in I_{j3}, \\ p_{j3}(\xi_j)e^{-2i\beta(\xi_j,\xi)}\left(z - \xi_j\right)^{-2i\epsilon_j\nu(\xi_j)}, & z \in \Sigma_{j3}, \end{cases} \tag{4.61}$$

$$f_{j4}(z) = \begin{cases} p_{j4}(z)\delta^{2}(z), & z \in I_{j4}, \\ p_{j4}(\xi_j)e^{2i\beta(\xi_j,\xi)}\left(z - \xi_j\right)^{2i\epsilon_j\nu(\xi_j)}, & z \in \Sigma_{j4}, \end{cases} \tag{4.62}$$

where

$$\begin{aligned} &I_{11} = I_{12} := (\xi_1, +\infty), \quad I_{21} = I_{22} := \left(\frac{\xi_2}{2}, \xi_2\right), \\ &I_{31} = I_{32} := \left(\xi_3, \frac{\xi_3}{2}\right), \quad I_{41} = I_{42} := (-\infty, \xi_4), \end{aligned} \tag{4.63}$$

$$\begin{aligned} &I_{13} = I_{14} := \left(\frac{\xi_2 + \xi_1}{2}, \xi_1\right), \quad I_{23} = I_{24} := \left(\xi_2, \frac{\xi_2 + \xi_1}{2}\right), \\ &I_{33} = I_{34} := \left(\frac{\xi_4 + \xi_3}{2}, \xi_3\right), \quad I_{43} = I_{44} := \left(\xi_4, \frac{\xi_4 + \xi_3}{2}\right). \end{aligned} \tag{4.64}$$

And f_{jk}, $j, k = 1, 2, 3, 4$ have following properties:

$$|\bar{\partial} f_{jk}(z)| \lesssim |p'_{jk}(\Re z)| + |z - \xi_j|^{-1/2}, \quad z \in \Omega_{jk}, \quad j, k = 1, 2, 3, 4, \tag{4.65}$$

$$|f_{jk}(z)| \lesssim \sin^2\left(\kappa_0 \arg\left(z - \xi_j\right)\right) + \langle \Re z \rangle^{-1}, \quad z \in \Omega_{jk}, \quad j, k = 1, 2, 3, 4. \tag{4.66}$$

Moreover, as $z \to 1$,

$$|\bar{\partial} f_{jk}(z)| \lesssim |p'_{jk}||z - 1|, \quad z \in \Omega_{24}, \Omega_{23}, \tag{4.67}$$

$$|\bar{\partial} f_{jk}(z)| \lesssim |p'_{jk}||z + 1|, \quad z \in \Omega_{34}, \Omega_{33}. \tag{4.68}$$

Proof. We take $f_{11}(z)$ and $f_{24}(z)$ as examples to present this proof. The continuous extension of $f_{11}(z)$ on Ω_{11} can be constructed by

$$
\begin{aligned}
f_{11}(z) = p_{11}(\xi_1) e^{-2i\beta(\xi_1,\xi)} (z-\xi_1)^{-2i\nu(\xi_1)} \left[1-\cos\left(\kappa_0 \arg\left(z-\xi_1\right)\right)\right] \\
+ p_{11}(\Re z)\delta_+^{-2}(z) \cos\left(\kappa_0 \arg\left(z-\xi_1\right)\right).
\end{aligned} \tag{4.69}
$$

Where $\kappa_0 = \frac{\pi}{2\phi}$. Denote $z = \xi_1 + le^{i\varphi} := \xi_1 + u + iv$, where $l, \varphi, u, v \in \mathbb{R}$. Firstly, we have $|p_{11}(\Re z)| = \frac{|r(\Re z)|}{1-|r(\Re z)|^2} \leqslant \frac{|r(\Re z)|}{1-\rho^2} \lesssim \langle \Re z \rangle^{-1}$. Recalling (4.7), we obtain (4.66). Applying $\bar{\partial} = \frac{1}{2} e^{i\varphi}(\partial_l + il^{-1}\partial_\varphi)$ to f_{11}, we have

$$
\begin{aligned}
\bar{\partial} f_{11} = \left[p_{11}\delta_+^{-2}(z) - p_{11}(\xi_1) e^{-2i\beta(\xi_1,\xi)} (z-\xi_1)^{-2i\nu(\xi_1)}\right] \bar{\partial} \cos\left(\kappa_0 \varphi\right) \\
+ \frac{1}{2}\delta_+^{-2}(z) p_{11}'(u,\xi) \cos\left(\kappa_0\varphi\right).
\end{aligned} \tag{4.70}
$$

Recalling (4.8), we get (4.65) at once.

For f_{24}, taking the same method to f_{11}, we have

$$
\begin{aligned}
\bar{\partial} f_{24} = \left[p_{24}\delta^{2}(z) - p_{24}(\xi_2) e^{2i\beta(\xi_2,\xi)} (z-\xi_2)^{-2i\nu(\xi_2)}\right] \bar{\partial} \cos\left(\kappa_0 \varphi\right) \\
+ \frac{1}{2}\delta^{2}(z) p_{24}'(u,\xi) \cos\left(\kappa_0\varphi\right).
\end{aligned} \tag{4.71}
$$

Finally z near 1, we have $\varphi \to 0$, thus we obtain

$$
|\bar{\partial} f_{24}| \lesssim |p_{24}'| \cos(\kappa_0\varphi) \lesssim |p_{24}'||z-1|. \tag{4.72}
$$

Estimation for the other f_{jk} could be given via similar techniques. □

Define the second transformation

$$
M^{(2)}(z) := M^{(2)}(x,t;z) = M^{(1)}(z) R^{(2)}(z), \tag{4.73}
$$

which constructs the mixed $\bar{\partial}$-RH problem as follows

RH problem 4.10. *Find a 2×2 matrix-valued function $M^{(2)}(x,t;z)$ such that*
- $M^{(2)}(z)$ is continuous in $\mathbb{C} \backslash \Sigma^{(2)}$, where $\Sigma^{(2)}$ is defined by (4.26).
- $M^{(2)}(z)$ takes continuous boundary values $M_\pm^{(2)}(z)$ on $\Sigma^{(2)}$ with jump relation

$$
M_+^{(2)}(z) = M_-^{(2)}(z) V^{(2)}(z), \tag{4.74}
$$

where

$$V^{(2)}(z)=\begin{cases} R^{(2)}(z)^{-1}|_{\Sigma_{j1}} & z\in\Sigma_{j1}, \quad j=0^{\pm},1,2,3,4,\\ R^{(2)}(z)^{-1}|_{\Sigma_{j4}} & z\in\Sigma_{j4}, \quad j=1,2,3,4,\\ R^{(2)}(z)|_{\Sigma_{j2}} & z\in\Sigma_{j2}, \quad j=0^{\pm},1,2,3,4,\\ R^{(2)}(z)|_{\Sigma_{j3}} & z\in\Sigma_{j3}, \quad j=1,2,3,4,\\ R^{(2)}(z)^{-1}|_{\Omega_{jk}^{\frac{1}{2}}}R^{(2)}(z)|_{\Omega_{lm}^{\frac{1}{2}}}, & z\in\Sigma_{n\pm}^{(1/2)}, \quad n=1,2,3,4. \end{cases} \tag{4.75}$$

- *Asymptotic behavior*

$$M^{(2)}(x,t;z)=I+\mathcal{O}(z^{-1}), \quad z\to\infty, \tag{4.76}$$

$$M^{(2)}(x,t;z)=\frac{\sigma_2}{z}+\mathcal{O}(1), \quad z\to 0. \tag{4.77}$$

- *For $z\in\mathbb{C}$, we have $\bar\partial$-derivative equality $\bar\partial M^{(2)}=M^{(2)}\bar\partial R^{(2)}$, where*

$$\bar\partial R^{(2)}=\begin{cases} \begin{pmatrix}1 & \bar\partial f_{j1}e^{2it\theta}\\ 0 & 1\end{pmatrix}, & z\in\Omega_{j1}, \quad j=0^{\pm},1,2,3,4,\\ \begin{pmatrix}1 & 0\\ \bar\partial f_{j2}e^{-2it\theta} & 1\end{pmatrix}, & z\in\Omega_{j2}, \quad j=0^{\pm},1,2,3,4,\\ \begin{pmatrix}1 & \bar\partial f_{j3}e^{2it\theta}\\ 0 & 1\end{pmatrix}, & z\in\Omega_{j3}, \quad j=1,2,3,4,\\ \begin{pmatrix}1 & 0\\ \bar\partial f_{j4}e^{-2it\theta} & 1\end{pmatrix}, & z\in\Omega_{j4}, \quad j=1,2,3,4,\\ 0, & elsewhere. \end{cases} \tag{4.78}$$

Remark 4.11. Notice the boundaries of f_{jk}, $j=0^{\pm}$, $k=1,2$ defined by (4.52), (4.53), we actually know that $V^{(2)}(z)=I$, for $z\in\Sigma_{0^{\pm}k}$, $k=1,2$.

Aiming at solving the mixed $\bar\partial$-RH problem 4.10, we decompose it to a pure RH problem for $M^{(PR)}$ with $\bar\partial R^{(2)}\equiv 0$ as well as a pure $\bar\partial$-problem $M^{(3)}$ with nonzero $\bar\partial R^{(2)}$ derivatives. This step can be shown as the following structure

$$M^{(2)}=M^{(3)}M^{(PR)}\begin{cases}\bar\partial R^{(2)}\equiv 0\to M^{(PR)},\\ \bar\partial R^{(2)}\neq 0\to M^{(3)}=M^{(2)}{M^{(PR)}}^{-1}.\end{cases} \tag{4.79}$$

4.4. *Analysis on pure RH problem*

In this subsection, we mainly focus on the analysis for pure RH problem $M^{(PR)}$, which include three parts: global parametrix, local parametrix as well as small norm RH problem. Noticing that $M^{(PR)}$ is a RH problem with $\bar\partial R^{(2)}\equiv 0$, thus, RH conditions for $M^{(PR)}$ are as follows.

RH problem 4.12. *Find a 2×2 matrix-valued function $M^{(PR)}(x,t;z)$ such that*
- $M^{(PR)}(z)$ is analytic in $\mathbb{C} \backslash \Sigma^{(2)}$.
- $M^{(PR)}(z)$ takes continuous boundary values $M_{\pm}^{(PR)}(z)$ on $\Sigma^{(2)}$ with jump relation

$$M_{+}^{(PR)}(z) = M_{-}^{(PR)}(z) V^{(2)}(z). \tag{4.80}$$

- Asymptotic behavior

$$M^{(PR)}(x,t;z) = I + \mathcal{O}(z^{-1}), \quad z \to \infty, \tag{4.81}$$

$$M^{(PR)}(x,t;z) = \frac{\sigma_2}{z} + \mathcal{O}(1), \quad z \to 0. \tag{4.82}$$

Define $U(\xi)$ as the union set of neighborhood of saddle point ξ_j for $j = 1,2,3,4$.

$$U(\xi) = \bigcup_{j=1,2,3,4} U_{\varrho}(\xi_j), \text{ with } U_{\varrho}(\xi_j) = \left\{z : |z - \xi_j| < \varrho\right\}, \tag{4.83}$$

where

$$\varrho < \frac{1}{3}\min\left\{\min\{|\Im\eta_n|\}_{n=1}^{2N}, \quad \min_{k\neq l}|\eta_l - \eta_k|, \quad \frac{1}{2}\min_{j=1,2,3,4}|\xi_j \pm 1| \quad \frac{1}{2}\min_{j=1,2,3,4}|\xi_j|\right\}. \tag{4.84}$$

Remark 4.13. The third and fourth restriction of (4.84) is to remove the singularity $z = 0, \pm 1$ from local model which will be discussed in Subsection 4.4.2.

Proposition 4.14. *For $1 \leqslant p \leqslant +\infty$, there exists a constant $\hbar = \hbar(p) > 0$, such that the jump matrix $V^{(2)}$ defined in (4.75) admits the following estimation as $t \to +\infty$*

$$\|V^{(2)} - I\|_{L^p(\Sigma_{jk}\backslash U_{\varrho}(\xi_j))} = \mathcal{O}(e^{-\hbar t}), \quad \text{for} \quad j,k = 1,2,3,4. \tag{4.85}$$

Proof. We take $z \in \Sigma_{24}\backslash U_{\varrho}(\xi_2)$ as an example, the other cases can be proved in a similar way. For $z \in \Sigma_{24}\backslash U_{\varrho}(\xi_2)$, $1 \leqslant p < +\infty$, by using (4.75) and (4.66), we have

$$\|V^{(2)} - I\|_{L^p(\Sigma_{24}\backslash U_{\varrho}(\xi_2))} = \|p_{24}(\xi_2) e^{2i\beta(\xi_2,\xi)} (z - \xi_2)^{-2i\nu(\xi_2)} e^{-2it\theta}\|_{L^p(\Sigma_{24}\backslash U_{\varrho}(\xi_2))} \tag{4.86}$$

$$\lesssim \|e^{-2it\theta}\|_{L^p(\Sigma_{24}\backslash U_{\varrho}(\xi_2))}. \tag{4.87}$$

For $z \in \Sigma_{24}\backslash U_{\varrho}(\xi_2)$, we still denote $z = \xi_2 + le^{i\varphi} = \xi_2 + u + iv$, $l > \varrho$. With the help of Proposition 4.6, we have

$$\begin{aligned} \|e^{-2it\theta}\|_{L^p(\Sigma_{24}\backslash U_{\varrho}(\xi_2))}^p &\lesssim \int_{\Sigma_{24}\backslash U_{\varrho}(\xi_2)} e^{-2tpc\left(1+|z|^{-2}\right)v^2} dz \\ (1 + |z|^{-2} \geqslant 1) &\lesssim \int_{\varrho}^{\infty} e^{-pctl} dl \\ &\lesssim t^{-1} e^{-pct\varrho}, \end{aligned} \tag{4.88}$$

where the value of $c = c(\xi)$ above changes from line to line. $\square$

Proposition 4.15. *For $1 \leqslant p < +\infty$, there exists a constant $\hbar' = \hbar'(p) > 0$, such that the jump matrix $V^{(2)}$ defined in (4.75) admits the following estimate as $t \to +\infty$*

$$\|V^{(2)} - I\|_{L^p(\Sigma_{j\pm}^{(1/2)})} = \mathcal{O}(e^{-\hbar' t}), \quad \text{for} \quad j = 1, 2, 3, 4. \tag{4.89}$$

Proof. We only give the details for $z \in \Sigma_{1+}^{(1/2)}$.

$$\begin{aligned}\|V^{(2)} - I\|_{L^p(\Sigma_{1+}^{(1/2)})} &= \|(f_{24} - f_{14})e^{-2it\theta}\|_{L^p(\Sigma_{1+}^{(1/2)})} \\ &\overset{(4.66)}{\lesssim} \|e^{-2it\theta}\|_{L^p(\Sigma_{1+}^{(1/2)})} \qquad (4.90) \\ &\lesssim t^{-\frac{1}{p}} e^{-ct}. \quad \square \qquad (4.91)\end{aligned}$$

4.4.1. Global parametrix: $M^{(\infty)}$

The leading order of $M^{(PR)}$ is approximated by a global parametrix (denoted by $M^{(\infty)}$) with exponentially decaying on the jump of $M^{(PR)}(z)$ (see Proposition 4.14 and Proposition 4.15). Thus we consider the following RH problem

RH problem 4.16. *Find a 2×2 matrix-valued function $M^{(\infty)}(x, t; z)$ which satisfies*
- $M^{(\infty)}(z)$ is analytical in $\mathbb{C}\backslash\{0\}$.
- Asymptotic behavior

$$M^{(\infty)}(x, t; z) = I + \mathcal{O}(z^{-1}), \quad z \to \infty, \tag{4.92}$$

$$M^{(\infty)}(x, t; z) = \frac{\sigma_2}{z} + \mathcal{O}(1), \quad z \to 0. \tag{4.93}$$

Then the following result is standard.

Proposition 4.17. *The unique solution of RH problem 4.16 is given by*

$$M^{(\infty)}(z) = I + \frac{\sigma_2}{z}. \tag{4.94}$$

4.4.2. Local parametrix near saddle points: $M^{(LC)}(z)$

The sub-leading contribution stems form the local behavior near critical points ξ_j, $j = 1, 2, 3, 4$. It turns out that the local parametrix (denoted by $M^{(LC,\xi_j)}$, $j = 1, 2, 3, 4$ below) can be constructed in terms of the solution of the well-known Webb (parabolic cylinder) equation. Denote $U_\varrho(\xi_j)$ the open disk of radius ϱ defined by (4.83) around ξ_j, $j = 1, 2, 3, 4$ respectively. And define the contours $\Sigma^{(LC)} := \left(\cup_{j,k=1,2,3,4}\Sigma_{jk}\right) \cap U(\xi)$ (see Fig. 7).

Now we turn to the following localized RH problem.

RH problem 4.18. *Find a 2×2 matrix-valued function $M^{(LC,\xi_j)}(x, t; z)$ such that*
- $M^{(LC,\xi_j)}(z)$ is analytical in $\mathbb{C}\backslash\Sigma^{(LC,\xi_j)}$, where $\Sigma^{(LC,\xi_j)} := \Sigma_{jk}^{(LC)}$, $k = 1, 2, 3, 4$.

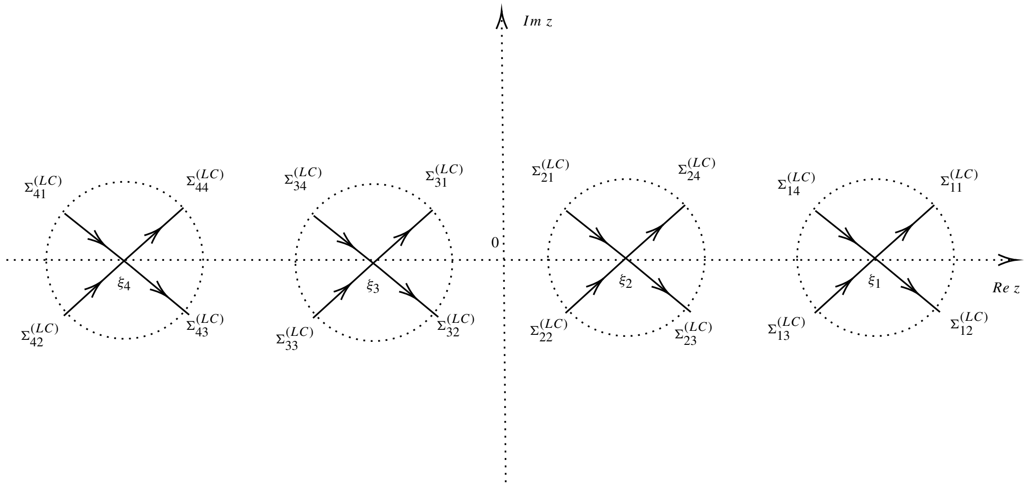


Fig. 7. Jump contour $\Sigma^{(LC)}$ of $M^{(LC,\,\xi_j)}(z)$, $j = 1, 2, 3, 4$.

- *$M^{(LC,\xi_j)}(z)$ takes continuous boundary values $M_{\pm}^{(LC,\xi_j)}(z)$ on $\Sigma^{(LC,\xi_j)}$ with jump relation*

$$M_{+}^{(LC,\xi_j)}(z) = M_{-}^{(LC,\xi_j)}(z)V^{(LC,\xi_j)}(z), \quad z \in \Sigma^{(LC,\xi_j)}, \tag{4.95}$$

where

$$V^{(LC,\xi_j)}(z) = \begin{cases} \begin{pmatrix} 1 & -\frac{\overline{r(\xi_j)}}{1-|r(\xi_j)|^2} e^{-2i\beta(\xi_j,\xi)}(z-\xi_j)^{-2i\epsilon_j\nu(\xi_j)}e^{2it\theta} \\ 0 & 1 \end{pmatrix}, & z \in \Sigma_{j1}^{(LC)}, \\ \begin{pmatrix} 1 & 0 \\ \frac{r(\xi_j)}{1-|r(\xi_j)|^2} e^{2i\beta(\xi_j,\xi)}(z-\xi_j)^{2i\epsilon_j\nu(\xi_j)}e^{-2it\theta} & 1 \end{pmatrix}, & z \in \Sigma_{j2}^{(LC)}, \\ \begin{pmatrix} 1 & -\overline{r(\xi_j)} e^{-2i\beta(\xi_j,\xi)}(z-\xi_j)^{-2i\epsilon_j\nu(\xi_j)}e^{2it\theta} \\ 0 & 1 \end{pmatrix}, & z \in \Sigma_{j3}^{(LC)}, \\ \begin{pmatrix} 1 & 0 \\ r(\xi_j) e^{2i\beta(\xi_j,\xi)}(z-\xi_j)^{2i\epsilon_j\nu(\xi_j)}e^{-2it\theta} & 1 \end{pmatrix}, & z \in \Sigma_{j4}^{(LC)}. \end{cases} \tag{4.96}$$

- *$M^{(LC,\xi_j)}(x,t;z) = I + \mathcal{O}(z^{-1}), \quad z \to \infty$ (Figs. 8 and 9).*

For z near ξ_j, $j = 1, 2, 3, 4$, we have

$$\theta(z) = \theta(\xi_j) + \frac{\theta''(\xi_j)}{2}(z-\xi_j)^2 + \mathcal{O}(|z-\xi_j|^3), \quad z \to \xi_j, \ j = 1, 2, 3, 4. \tag{4.97}$$

Thus, for $z \in U_{\varrho}(\xi_j)$, we define the rescaled variable ζ by

$$\zeta(z) = \left(2t\epsilon_j\theta''\left(\xi_j\right)\right)^{\frac{1}{2}}\left(z-\xi_j\right), \ j = 1, 2, 3, 4, \tag{4.98}$$

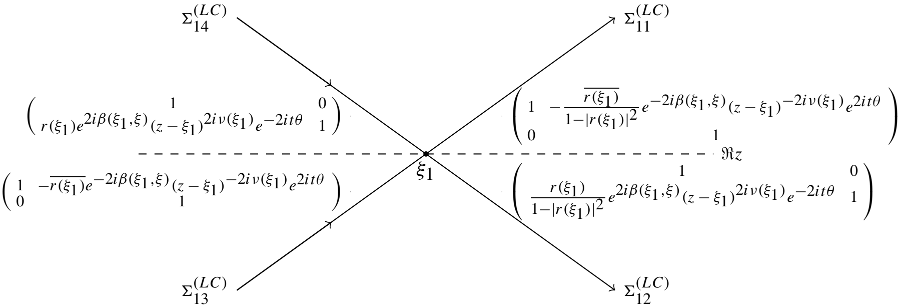


Fig. 8. The contour $\Sigma^{(LC,\xi_1)}$ and the jump matrix on it.

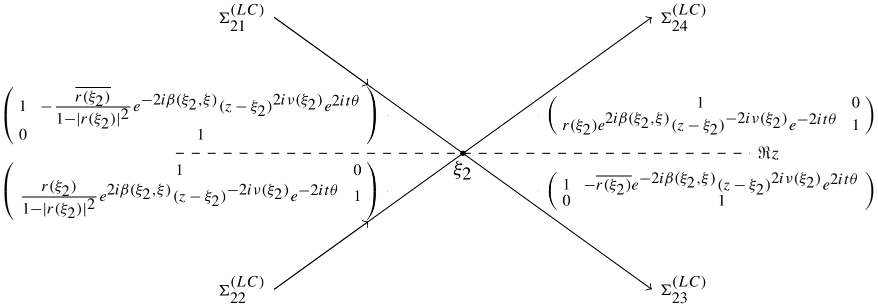


Fig. 9. The contour $\Sigma^{(LC,\xi_2)}$ and the jump matrix on it.

which is to match the standard model presented in Appendix A.

And the scaling operator N_{ξ_j} admits the following mapping

$$N_{\xi_j} : U_{\varrho}(\xi_j) \longrightarrow U_0, \;\; j=1,2,3,4, \tag{4.99}$$

$$z \longmapsto \zeta, \tag{4.100}$$

where U_0 is a neighborhood of $\zeta = 0$.

Choose the free variable r_{ξ_j} appeared in the Appendix A by

$$r_{\xi_j} = r(\xi_j)e^{2i\beta(\xi_j,\xi)-2it\theta(\xi_j)}\exp\left[-i\epsilon_j\nu(\xi_j)\log(2t\epsilon_j\theta''\left(\xi_j\right))\right], \tag{4.101}$$

with the equality $|r(\xi_j)|^2 = |r_{\xi_j}|^2$, $j=1,2,3,4$.

In the above expression, the complex powers are defined by choosing the branch of the logarithm with $0 < \arg\zeta < 2\pi$ near ξ_1, ξ_3, and the branch of the logarithm with $-\pi < \arg\zeta < \pi$ near ξ_2, ξ_4. Through this scaling of variable, the jump $V^{(LC,\xi_j)}(z)$ can be approximated by the jump of a parabolic cylinder model which is shown in Appendix A.

Remark 4.19. In the expansion (4.97), the higher order term as $z \to \xi_j$ could be ignored under the condition $|x/t| = \mathcal{O}(1)$. Without loss of generality, we take the neighborhood of ξ_1 as an example, we expand

$$\theta(z)=\theta(\xi_1)+\frac{\theta''(\xi_1)}{2}(z-\xi_1)^2+\theta_c(z-\xi_1)^3, \tag{4.102}$$

where $\theta_c=\frac{\theta'''(\kappa\xi_1+(1-\kappa)z)}{3!}$, $\kappa\in(0,1)$ is the coefficient of remainder.

Owe to the scaling (4.98), we have the following transformation

$$N:g\longrightarrow (Ng)(\zeta):=g\left((2t\theta''(\xi_1))^{-\frac{1}{2}}\zeta+\xi_1\right), \tag{4.103}$$

which acts on $e^{2it\theta(z)}$ to obtain

$$e^{2it\theta(z)}=e^{2it(N\theta)(\zeta)}\overset{(4.98)}{=}e^{2it\theta(\xi_1)}\cdot e^{\frac{i}{2}\zeta^2}\cdot e^{2it\theta_c\cdot(2t\theta''(\xi_1))^{-\frac{3}{2}}\zeta^3}. \tag{4.104}$$

The first and second term of R.H.S. of (4.104) are used to match parabolic cylinder model. What we care about is the third term. Since $\zeta\in U_0$, the neighborhood of zero, we can set $\zeta=u+iv$, $|u|<\varepsilon$, $|v|<\varepsilon$. Thus

$$\begin{aligned}|e^{2it\theta_c(2t\theta''(\xi_1))^{-\frac{3}{2}}\zeta^3}|&=|e^{2it\theta_c\cdot(2t\theta''(\xi_1))^{-\frac{3}{2}}(u+iv)^3}|\\&=\exp\left((2t\theta''(\xi_1))^{-\frac{3}{2}}\Re\left(2it\left(\Re\left(\theta_c\right)+i\Im\left(\theta_c\right)\right)\cdot(u+iv)^3\right)\right)\\&=\exp\left[-(2t)^{-\frac{1}{2}}(\theta''(\xi_1))^{-\frac{3}{2}}\left(\Re(\theta_c)\left(3u^2v-v^3\right)+\Im(\theta_c)\left(u^3-3uv^2\right)\right)\right]\\&\to 1\quad\text{as}\quad t\to+\infty,\end{aligned} \tag{4.105}$$

by which the effects of the higher power could be ignored. The premise of this result is that $\Re(\theta_c)\left(3u^2v-v^3\right)+\Im(\theta_c)\left(u^3-3uv^2\right)$ is finite, which follows from finite ξ.

As a consequence, a standard local parametrix for $M^{(LC,\xi_j)}$, $j=1,2,3,4$ is constructed by

$$M^{(LC,\xi_j)}(x,t;z)=M^{(\infty)}\cdot M^{(PC,\xi_j)}\left(\xi,\zeta(z)=\left(2t\epsilon_j\theta''\left(\xi_j\right)\right)^{\frac{1}{2}}\left(z-\xi_j\right)\right), \tag{4.106}$$

where

$$M^{(PC,\xi_j)}\left(\xi,\zeta(z)\right)=I+\frac{\left(2t\epsilon_j\theta''(\xi_j)\right)^{-\frac{1}{2}}}{z-\xi_j}\begin{pmatrix}0&\epsilon_j i\beta_{12}^{(\xi_j)}\\-\epsilon_j i\beta_{21}^{(\xi_j)}&0\end{pmatrix}+\mathcal{O}(\zeta^{-2}), \tag{4.107}$$

where $\beta_{12}^{(\xi_j)}$ and $\beta_{21}^{(\xi_j)}$ are defined by (A.24)-(A.26) for $j=1,3$, while are defined by (A.50)-(A.52) for $j=2,4$.

Remark 4.20. Here we directly calculate $M^{(LC,\xi_j)}(z)$, $j=1,3$ and $M^{(LC,\xi_j)}(z)$, $j=2,4$, because the original circular symmetry reduction $M(z)=\mp z^{-1}M(z^{-1})\sigma_2$ is destroyed in these local models.

Now we consider a new RH problem $M^{(LC)}(x,t;z)$, which include the contribution from $M^{(LC,\xi_j)}$, $j=1,2,3,4$.

RH problem 4.21. *Find a 2×2 matrix-valued function $M^{(LC)}(z)$ such that*
- $M^{(LC)}(z)$ is analytical off $\Sigma^{(LC)}$.
- $M^{(LC)}(z)$ takes continuous boundary values $M_{\pm}^{(LC)}(z)$ on $\Sigma^{(LC)}$ with jump relation

$$M_{+}^{(LC)}(z)=M_{-}^{(LC)}(z)V^{(LC)}(z), \quad z\in\Sigma^{(LC)}, \tag{4.108}$$

where $V^{(LC)}(z)=V^{(2)}(z)$.
- $M^{(LC)}(x,t;z)=I+\mathcal{O}(z^{-1}), \quad z\to\infty$.

$V^{(LC)}(z)$ admits a factorization

$$V^{(LC)}(z)=\left(I-w_{jk}^{-}\right)^{-1}\left(I+w_{jk}^{+}\right), \tag{4.109}$$

where

$$w_{jk}^{-}=0, \quad w_{jk}^{+}=V^{(LC)}-I, \tag{4.110}$$

and the superscript $\pm$ indicate the analyticity in the positive and negative neighborhood of the contour respectively.

Recall the Cauchy projection operator $C_{\pm}$ on $\Sigma_{jk}^{(LC)}$, $j,k=1,2,3,4$

$$C_{\pm}f(z)=\lim_{s\to z,\ z\in\Sigma_{jk,\pm}^{(LC)}}\frac{1}{2\pi i}\int_{\Sigma_{jk}^{(LC)}}\frac{f(s)}{s-z}ds. \tag{4.111}$$

Define the following operator on $\Sigma_{jk}^{(LC)}$, $k=1,2,3,4$ as follows

$$C_{w_{jk}}(f):=C_{-}(fw_{jk}^{+}). \tag{4.112}$$

Then we review some notations as follows

$$w_j=\sum_{k=1}^{4}w_{jk}, \quad \Sigma^{(LC,\xi_j)}=\bigcup_{k=1,2,3,4}\Sigma_{jk}^{(LC)}, \tag{4.113}$$

$$w=\sum_{j=1}^{4}w_j=\sum_{j,k=1}^{4}w_{jk}, \quad \Sigma^{(LC)}=\bigcup_{j=1,2,3,4}\Sigma^{(LC,\xi_j)}=\bigcup_{j,k=1,2,3,4}\Sigma_{jk}^{(LC)}. \tag{4.114}$$

It is obvious to find that $C_w=\sum_{j=1}^{4}C_{w_j}=\sum_{j,k=1}^{4}C_{w_{jk}}$. A direct calculation shows that $\|w_{jk}\|_{L^2(\Sigma_{jk}^{(LC)})}=\mathcal{O}(t^{-1/2})$, which implies that $1-C_w$, $1-C_{w_j}$ and $1-C_{w_{jk}}$ exist as $t\to+\infty$.

Via standard Beals-Coifman theory [3], we know that $M^{(LC)}$ can be uniquely shown by

$$M^{(LC)} = I + C(\mu w), \tag{4.115}$$

where $\mu \in I + L^2(\Sigma^{(LC)})$ is the solution of the singular integral equation $\mu = I + C_w(\mu)$. $M^{(LC)}(z)$ could be expressed in terms of the following integral.

$$M^{(LC)} = I + \frac{1}{2\pi i} \int_{\Sigma^{(LC)}} \frac{(1-C_w)^{-1} I w}{s-z} ds. \tag{4.116}$$

Proposition 4.22. *As $t \to +\infty$, for $j \neq k$*

$$\|C_{w_j} C_{w_k}\|_{L^2(\Sigma^{(LC)})} = \mathcal{O}(t^{-1}), \quad \|C_{w_j} C_{w_k}\|_{L^\infty(\Sigma^{(LC)}) \to L^2(\Sigma^{(LC)})} = \mathcal{O}(t^{-1}). \tag{4.117}$$

Proof. Thanks to the observation of Varzugin [29], we have

$$1 - \sum_{j \neq k} C_{w_j} C_{w_k} \left(1 - C_{w_k}\right)^{-1} = (1 - C_w) \left(1 + \sum_{j=1}^{4} C_{w_j} \left(1 - C_{w_j}\right)^{-1}\right), \tag{4.118}$$

$$1 - \sum_{j \neq k} \left(1 - C_{w_k}\right)^{-1} C_{w_j} C_{w_k} = \left(1 + \sum_{j=1}^{4} C_{w_j} \left(1 - C_{w_j}\right)^{-1}\right) (1 - C_w). \tag{4.119}$$

The result follows from $\|w_{jk}\|_{L^2(\Sigma_{jk}^{(LC)})} = \mathcal{O}(t^{-1/2})$. $\square$

The following proposition reveals that the contribution for $M^{(LC)}(z)$ could be separated by each $M^{(LC,\xi_j)}(z)$, $j = 1, 2, 3, 4$.

Proposition 4.23. *As $t \to +\infty$,*

$$\int_{\Sigma^{(LC)}} \frac{(1-C_w)^{-1} I w}{s-z} = \sum_{j=1}^{4} \int_{\Sigma^{(LC,\xi_j)}} \frac{\left(1 - C_{w_j}\right)^{-1} I w_j}{s-z} + \mathcal{O}(t^{-1}). \tag{4.120}$$

Proof. Decompose the resolvent $(1 - C_w)^{-1} I$ as

$$(1 - C_w)^{-1} I = I + \sum_{j=1}^{4} C_{w_j} (1 - C_{w_j})^{-1} I + QPRI, \tag{4.121}$$

where

$$Q := 1 + \sum_{j=1}^{4} C_{w_j}(1 - C_{w_j})^{-1}, \tag{4.122}$$

$$P := \left(1 - \sum_{j \neq k} C_{w_j} C_{w_k} \left(1 - C_{w_k}\right)^{-1}\right)^{-1}, \tag{4.123}$$

$$R := \sum_{j \neq k} C_{w_j} C_{w_k} (1 - C_{w_k})^{-1}. \tag{4.124}$$

Using Cauchy-Schwarz inequality and Proposition 4.23, we have

$$\begin{aligned} \left| \int QPRIw \right| &\leqslant \|Q\|_{L^2\left(\Sigma^{(LC,\xi_j)}\right)} \|P\|_{L^2\left(\Sigma^{\left(LC,\xi_j\right)}\right)} \|R\|_{L^2\left(\Sigma^{\left(LC,\xi_j\right)}\right)} \|w\|_{L^2} \\ &\lesssim t^{-1}. \quad \square \end{aligned} \tag{4.125}$$

Combining (4.107), the following for $M^{(LC)}(z)$ proposition follows.

Proposition 4.24. *As $t \to +\infty$, we have*

$$M^{(LC)}(z) = M^{(\infty)}(z) \cdot M^{(PC)}(z), \tag{4.126}$$

where

$$M^{(PC)}(z) = I + t^{-\frac{1}{2}} \sum_{j=1}^{4} \frac{i\epsilon_j A_j^{mat}}{\left(2\epsilon_j \theta''(\xi_j)\right)^{\frac{1}{2}} \left(z - \xi_j\right)} + \mathcal{O}(t^{-1}), \tag{4.127}$$

with

$$A_j^{mat} = \begin{pmatrix} 0 & \beta_{12}^{(\xi_j)} \\ -\beta_{21}^{(\xi_j)} & 0 \end{pmatrix} \tag{4.128}$$

4.4.3. Error: a small norm RH problem

Define the error matrix $M^{(err)}$ by

$$M^{(err)}(z) = \begin{cases} M^{(PR)}(z) M^{(\infty)}(z)^{-1}, & z \in \mathbb{C} \backslash U(\xi), \\ M^{(PR)}(z) M^{(LC)}(z)^{-1}, & z \in U(\xi). \end{cases} \tag{4.129}$$

RH conditions for $M^{(err)}$ are as follows.

RH problem 4.25. *Find a 2×2 matrix-valued function $M^{(err)}(z)$ such that*
- $M^{(err)}$ is analytical in $\mathbb{C} \backslash \Sigma^{(err)}$, where

$$\Sigma^{(err)} := \partial U(\xi) \cup \left(\Sigma^{(2)} \backslash U(\xi)\right); \tag{4.130}$$

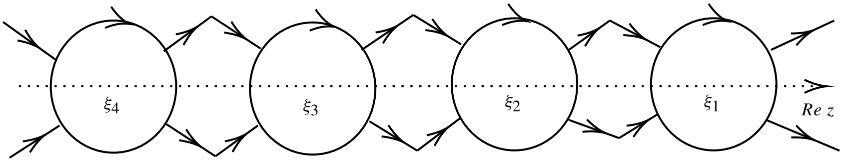


Fig. 10. Jump contour of $M^{(err)}(z)$.

- $M^{(err)}$ takes continuous boundary values $M_{\pm}^{(err)}(z)$ on $\Sigma^{(err)}$ and

$$M_{+}^{(err)}(z) = M_{-}^{(err)}(z)V^{(err)}(z), \tag{4.131}$$

where

$$V^{(err)}(z) = \begin{cases} M^{(\infty)}(z)V^{(2)}(z)M^{(\infty)}(z)^{-1}, & z \in \Sigma^{(2)} \backslash U(\xi), \\ M^{(\infty)}(z)M^{(PC)}(z)M^{(\infty)}(z)^{-1}, & z \in \partial U(\xi). \end{cases} \tag{4.132}$$

- $M^{(err)} = I + \mathcal{O}(z^{-1}), \quad z \to \infty$ (Fig. 10).

Taking into account Proposition 4.14 and Proposition 4.15, we can know that $V^{(err)}(z)$ exponentially decay to I for $z \in \Sigma^{(2)} \backslash U(\xi)$ and $z \in \Sigma_{j\pm}^{(1/2)}$, $j = 1, 2, 3, 4$. For $z \in \partial U(\xi)$, $M^{(\infty)}(z)$ is bounded, we obtain that

$$\begin{aligned} |V^{(err)} - I| &= |M^{(\infty)}(z)M^{(PC)}(z)M^{(\infty)}(z)^{-1} - I| \\ &= |M^{(\infty)}(z)(M^{(PC)}(z) - I)M^{(\infty)}(z)^{-1}| \\ &\overset{(4.127)}{=} \mathcal{O}(t^{-\frac{1}{2}}). \end{aligned} \tag{4.133}$$

According to Beals-Coifman theory, the solution for $M^{(err)}$ can be given by

$$M^{(err)} = I + \frac{1}{2\pi i} \int_{\Sigma^{(err)}} \frac{(I + \mu(s))(V^{(err)}(s) - I)}{s - z} ds, \tag{4.134}$$

where $\mu \in L^2(\Sigma^{(err)})$ is the unique solution of $(1 - C_{V^{(err)}})\mu = C_{V^{(err)}} I$. And $C_{V^{(err)}}$: $L^2(\Sigma^{(err)}) \to L^2(\Sigma^{(err)})$ is the Cauchy projection operator on $\Sigma^{(err)}$:

$$C_{V^{(err)}}(f)(z) = C_{-} f(V^{(err)} - I) = \lim_{s \to z,\ z \in \Sigma^{(err)}} \int_{\Sigma^{(err)}} \frac{f(s)(V^{(err)}(s) - I)}{s - z} ds. \tag{4.135}$$

Existence and uniqueness of μ follows from the boundedness of the Cauchy projection operator C_{-}, which implies

$$\|C_{V^{(err)}}\| \leqslant \|C_{-}\|_{L^2(\Sigma^{(err)}) \to L^2(\Sigma^{(err)})} \|V^{err} - I\|_{L^2(\Sigma^{(err)})} = \mathcal{O}(t^{-\frac{1}{2}}). \tag{4.136}$$

Moreover,

$$\|\mu\|_{L^2\Sigma^{(err)}} \lesssim \frac{\|C_{V^{(err)}}\|}{1-\|C_{V^{(err)}}\|} \lesssim t^{-\frac{1}{2}}. \tag{4.137}$$

Now we present the following proposition, which is helpful for the last asymptotics.

Proposition 4.26. *As $t \to \infty$, we have*

$$\begin{aligned} M_1^{(err)} = & \, t^{-1/2} \sum_{j=1}^{4} i\epsilon_j \left(2\epsilon_j \theta''(\xi_j)\right)^{-\frac{1}{2}} \left(1-\xi_j^{-2}\right)^{-1} \\ & \times \begin{pmatrix} -\frac{i}{\xi_j}\left(\beta_{12}^{(\xi_j)} - \beta_{21}^{(\xi_j)}\right) & \beta_{12}^{(\xi_j)} - \frac{1}{\xi_j^2}\beta_{21}^{(\xi_j)} \\ \frac{1}{\xi_j^2}\beta_{12}^{(\xi_j)} - \beta_{21}^{(\xi_j)} & \frac{i}{\xi_j}\left(\beta_{12}^{(\xi_j)} - \beta_{21}^{(\xi_j)}\right) \end{pmatrix} + \mathcal{O}(t^{-1}), \end{aligned} \tag{4.138}$$

where $M_1^{(err)}$ comes from the asymptotic expansion of $M^{(err)}$ as $z \to \infty$

$$M^{(err)} = I + z^{-1} M_1^{(err)} + \mathcal{O}(z^{-2}). \tag{4.139}$$

Moreover, the $(1,2)$-entry of $M_1^{(err)}$, denoted by $\left(M_1^{(err)}\right)_{12}$, is given by

$$\left(M_1^{(err)}\right)_{12} = t^{-1/2} \sum_{j=1}^{4} i\epsilon_j \left(2\epsilon_j \theta''(\xi_j)\right)^{-\frac{1}{2}} \left(1-\xi_j^{-2}\right)^{-1} \cdot \left(\beta_{12}^{(\xi_j)} - \frac{1}{\xi_j^2}\beta_{21}^{(\xi_j)}\right) + \mathcal{O}(t^{-1}). \tag{4.140}$$

Proof. Recalling (4.134), we know that

$$M_1^{(err)} = -\frac{1}{2\pi i} \int_{\Sigma^{(err)}} (I+\mu(s))\,(V^{(err)}(s)-I)ds := I_1 + I_2 + I_3, \tag{4.141}$$

where

$$I_1 = -\frac{1}{2\pi i} \oint_{\partial U(\xi)} \left(V^{err}(s) - I\right) ds, \tag{4.142}$$

$$I_2 = -\frac{1}{2\pi i} \int_{\Sigma^{(err)} \setminus U(\xi)} \left(V^{err}(s) - I\right) ds, \tag{4.143}$$

$$I_3 = -\frac{1}{2\pi i} \int_{\Sigma^{(err)}} \mu(s)(V^{err}(s) - I)ds. \tag{4.144}$$

Using Proposition 4.14 and Proposition 4.15, we obtain $|I_2| = \mathcal{O}(t^{-1})$.

Using (4.137) and (4.133), we obtain

$$|I_3| \lesssim \|\mu\|_{L^2}\|V^{err} - I\|_{L^2} \lesssim t^{-1}. \tag{4.145}$$

Finally, we deal with I_1

$$\begin{aligned}
I_1 &= -\frac{1}{2\pi i}\oint_{\partial U(\xi)} M^{(\infty)}(s)\left(M^{(PC)}(s) - I\right)M^{(\infty)}(s)^{-1}ds \\
&\overset{(4.127)}{=} -\frac{1}{2\pi i}\sum_{j=1}^{4}\oint_{\partial U_\varrho(\xi_j)} \frac{i\epsilon_j}{(2t\epsilon_j\theta''(\xi_j))^{\frac{1}{2}}(z-\xi_j)} M^{(\infty)}(s)A_j^{mat}(s)M^{(\infty)}(s)^{-1}ds \\
&\quad + \mathcal{O}(t^{-1}) \\
\text{(residue theorem)} &= t^{-1/2}\sum_{j=1}^{4} i\epsilon_j\left(2\epsilon_j\theta''(\xi_j)\right)^{-\frac{1}{2}} M^{(\infty)}(\xi_j)A_j^{mat}M^{(\infty)}(\xi_j)^{-1} + \mathcal{O}(t^{-1}) \\
&= t^{-1/2}\sum_{j=1}^{4} i\epsilon_j\left(2\epsilon_j\theta''(\xi_j)\right)^{-\frac{1}{2}}\left(1-\xi_j^{-2}\right)^{-1}\left(I + \frac{\sigma_2}{\xi_j}\right)A_j^{mat}\left(I - \frac{\sigma_2}{\xi_j}\right) \\
&\quad + \mathcal{O}(t^{-1}) \\
&= t^{-1/2}\sum_{j=1}^{4} i\epsilon_j\left(2\epsilon_j\theta''(\xi_j)\right)^{-\frac{1}{2}}\left(1-\xi_j^{-2}\right)^{-1} \\
&\quad \times \begin{pmatrix} -\frac{i}{\xi_j}\left(\beta_{12}^{(\xi_j)} - \beta_{21}^{(\xi_j)}\right) & \beta_{12}^{(\xi_j)} - \frac{1}{\xi_j^2}\beta_{21}^{(\xi_j)} \\ \frac{1}{\xi_j^2}\beta_{12}^{(\xi_j)} - \beta_{21}^{(\xi_j)} & \frac{i}{\xi_j}\left(\beta_{12}^{(\xi_j)} - \beta_{21}^{(\xi_j)}\right) \end{pmatrix} \\
&\quad + \mathcal{O}(t^{-1})
\end{aligned} \tag{4.146}$$

Summarizing I_1, I_2 and I_3, we obtain (4.138). $(M_1^{(err)})_{12}$ follows immediately. □

4.5. *Analysis on pure $\bar{\partial}$ problem*

Define

$$M^{(3)}(z) = M^{(2)}(z)M^{(PR)}(z)^{-1}. \tag{4.147}$$

Then $M^{(3)}$ satisfies the following $\bar{\partial}$ problem.

$\bar{\partial}$-Problem 4.27. *Find a 2×2 matrix-valued function $M^{(3)}(z)$ such that*
- *$M^{(3)}(z)$ is continuous in $\mathbb{C}$ and analytic in $\mathbb{C}\backslash\overline{\Omega}$.*
- *$M^{(3)}(z) = I + \mathcal{O}(z^{-1}), \quad z \to \infty$.*
- *For $z \in \mathbb{C}$, $\bar{\partial}M^{(3)}(z) = M^{(3)}(z)W^{(3)}(z)$ with $W^{(3)} = M^{(PR)}(z)\bar{\partial}R^{(2)}(z)M^{(PR)}(z)^{-1}$.*

Proof. It's enough to prove the following claims.

$M^{(3)}$ has no jumps. Indeed, since $M^{(2)}$ and $M^{(PR)}$ take the same jump matrix, we have

$$\begin{aligned} {M_-^{(3)}(z)}^{-1} M_+^{(3)}(z) &= M_-^{(PR)} (M_-^{(2)})^{-1} M_+^{(2)} (M_+^{(PR)})^{-1} \\ &= M_-^{(PR)} V^{(2)} (M_+^{(PR)})^{-1} = I. \end{aligned} \tag{4.148}$$

$M^{(3)}$ has no singularity at $z=0$. Near $z=0$, we have

$$(M^{(PR)})^{-1} = \frac{\sigma_2 (M^{(PR)})^{\mathrm{T}} \sigma_2}{1-z^{-2}}. \tag{4.149}$$

Thus

$$\lim_{z\to 0} M^{(3)} = \lim_{z\to 0} \frac{M^{(2)} \sigma_2 (M^{(PR)})^{\mathrm{T}} \sigma_2}{1-z^{-2}} = \mathcal{O}(1). \tag{4.150}$$

$M^{(3)}$ has no singularities at $z=\pm 1$. Indeed, as $z\to\pm 1$,

$$V(z) = \begin{pmatrix} c & \mp i c \\ \pm i\bar{c} & \bar{c} \end{pmatrix}, \quad {M^{(PR)}(z)}^{-1} = \frac{\pm 1}{2(z\mp 1)} \sigma_1 \begin{pmatrix} \gamma & \pm i\bar{\gamma} \\ \mp i\gamma & \bar{\gamma} \end{pmatrix} \sigma_1 + \mathcal{O}(1), \tag{4.151}$$

for some constants c and γ. Thus we have $\lim_{z\to\pm 1} M^{(3)}(z) = \mathcal{O}(1)$. $\square$

The solution of $\bar{\partial}$-Problem 4.27 can be solved by the following integral equation

$$M^{(3)}(z) = I - \frac{1}{\pi} \iint_{\mathbb{C}} \frac{M^{(3)}(s) W^{(3)}(s)}{s-z} dA(s), \tag{4.152}$$

where $A(s)$ is the Lebesgue measure on $\mathbb{C}$. Denote S be the Cauchy-Green integral operator

$$S[f](z) = -\frac{1}{\pi} \iint \frac{f(s) W^{(3)}(s)}{s-z} dA(s), \tag{4.153}$$

then (4.152) can be written as the following operator-valued equation

$$(1-S) M^{(3)}(z) = I. \tag{4.154}$$

To prove the existence of the operator at large time, we present the following proposition.

Proposition 4.28. *Consider the operator S defined by (4.153), then we have $S : L^\infty(\mathbb{C}) \to L^\infty(\mathbb{C}) \cap C^0(\mathbb{C})$ and*

$$\|S\|_{L^\infty(\mathbb{C})\to L^\infty(\mathbb{C})} \lesssim t^{-\frac{1}{4}}. \tag{4.155}$$

Proof. For any $f \in L^\infty$, we have

$$\|Sf\|_{L^\infty} \leqslant \|f\|_{L^\infty} \frac{1}{\pi} \iint\limits_{\mathbb{C}} \frac{|W^{(3)}(s)|}{|s-z|} dA(s). \tag{4.156}$$

Recalling the definition $W^{(3)} = M^{(PR)}(z)\bar{\partial}R^{(2)}(z)(M^{(PR)}(z))^{-1}$ and $\bar{\partial}R^{(2)}$, we know that $W^{(3)}(z) \equiv 0$ for $z \in \mathbb{C}\backslash\overline{\Omega}$. Besides, we only take into account that matrix-valued functions have support in sector Ω_{jk}. Based on these conditions, what we need to do is to control the boundedness of the integral $\iint_{\mathbb{C}} \frac{|W^{(3)}(s)|}{|s-z|} dA(s)$ for $z \in \Omega_{jk}$, $j = 0^\pm, 1, 2, 3, 4$, $k = 1, 2, 3, 4$. We present details for $z \in \Omega_{0^+1}$, $z \in \Omega_{24}$ and $z \in \Omega_{11}$, the proofs for the rest regions are similar.

Since $\det M^{(PR)}(z) = 1 - z^{-2}$ and $M^{(PR)}(z)^{-1} = (1-z^{-2})^{-1}\sigma_2(M^{(PR)})^{\mathrm{T}}\sigma_2$, we have

$$|W^{(3)}(s)| \leqslant |M^{(PR)}(s)|^2 |1-s^{-2}|^{-1} |\bar{\partial}R^{(2)}(s)|. \tag{4.157}$$

Next we estimate $M^{(PR)}$ as follows

$$|M^{(PR)}(s)| \lesssim 1 + |s|^{-1} \lesssim c\sqrt{(1+|s|^{-1})^2} \lesssim \sqrt{1+|s|^{-2}} = |s|^{-1}\sqrt{1+|s|^2} = |s|^{-1}\langle s\rangle. \tag{4.158}$$

Since $z = 1 \in \Omega_{24}$, we have

$$\frac{|s|^{-2}\langle s\rangle^2}{|1-s^{-2}|} = \frac{\langle s\rangle^2}{|1-s^2|} = \begin{cases} \mathcal{O}(1), & z \in \Omega_{0^+1}, \Omega_{11}, \\ \dfrac{\langle s\rangle}{|s-1|}, & z \in \Omega_{24}. \end{cases} \tag{4.159}$$

For $z \in \Omega_{0^+1}$ and Ω_{11}, the estimations are similar to [12]. However, for $z \in \Omega_{24}$, the singularities at $z = \pm 1$ should be treated in a more delicate way. To some extent, how to deal with the singularity points plays a core role in our analysis.

Introduce an inequality which plays an vital role in our analysis. Making $s = z_0 + le^{i\phi} = z_0 + u + iv$, $z = x + iy$, $u, v, x, y > 0$, we have

$$\begin{aligned} \left\| \frac{1}{s-z} \right\|_{L^q(v,\infty)} &\lesssim \left(\int\limits_{\mathbb{R}_+} \left[1 + \left(\frac{u+z_0-x}{v-y} \right)^2 \right]^{-\frac{q}{2}} (v-y)^{-q} du \right)^{\frac{1}{q}} \\ &= |v-y|^{\frac{1}{q}-1} \left(\int\limits_{\mathbb{R}_+} \left[1 + \left(\frac{u+z_0-x}{v-y} \right)^2 \right]^{-\frac{q}{2}} d\left(\frac{u+z_0-x}{v-y} \right) \right)^{1/q} \\ &\overset{q>1}{\lesssim} |v-y|^{1/q-1}. \end{aligned} \tag{4.160}$$

For $z \in \Omega_{0^+1}$, we make $z = x + iy$, $s = 0 + u + iv$. Thanks to (4.159), we have

$$\frac{1}{\pi} \iint\limits_{\Omega_{0^+1}} \frac{|W^{(3)}(s)|}{|s-z|} dA(s) \lesssim \frac{1}{\pi} \iint\limits_{\Omega_{0^+1}} \frac{|\bar{\partial} f_{0^+1} e^{2it\theta}|}{|s-z|} dA(s) = \frac{1}{\pi} \iint\limits_{\Omega_{0^+1}} \frac{|\bar{\partial} f_{0^+1}| e^{-2t\Im\theta}}{|s-z|} dA(s). \tag{4.161}$$

Recalling Proposition 4.8, we can divide the integral into two parts

$$\frac{1}{\pi}\iint_{\Omega_{0+1}}\frac{|\bar{\partial} f_{0+1}|e^{-2t\Im\theta}}{|s-z|}dA(s)\lesssim I_1+I_2, \tag{4.162}$$

where

$$I_1=\iint_{\Omega_{0+1}}\frac{|p'_{0+1}(s)|e^{-2t\Im\theta}}{|s-z|}dA(s),\quad I_2=\iint_{\Omega_{0+1}}\frac{|s|^{-\frac{1}{2}}e^{-2t\Im\theta}}{|s-z|}dA(s). \tag{4.163}$$

Notice that Ω_{0+1} is a bounded area, $0<x,u<\xi_2/2$ and $0<y,v<\frac{\xi_2\tan\phi}{2}$ $(<\frac{\xi_2}{2})$. Thus

$$\begin{aligned} I_1 &\overset{Cor\ 4.5}{\leqslant}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\frac{|p'_{0+1}|}{|s-z|}e^{-ctv}dudv\\ &\lesssim\int_{\mathbb{R}_+}\left\|\frac{1}{s-z}\right\|_{L^2(\mathbb{R}_+)}\|p'_{0+1}\|_{L^2(\mathbb{R}_+)}e^{-ctv}dv\\ &\lesssim\int_{\mathbb{R}_+}|v-y|^{-1/2}e^{-ctv}dv\\ &\lesssim t^{-\frac{1}{2}}. \end{aligned} \tag{4.164}$$

Next, we introduce the following inequality for $p>2$

$$\begin{aligned} \left\||s|^{-\frac{1}{2}}\right\|_{L^p(v,+\infty)} &=\left(\int_v^{+\infty}\left(\sqrt{u^2+v^2}\right)^{-p/2}du\right)^{1/p}\\ &\overset{l^2=u^2+v^2}{=}\left(\int_v^{+\infty}l^{-p/2}\cdot l\cdot u^{-1}dl\right)^{\frac{1}{p}}\\ &\lesssim v^{-\frac{1}{2}+\frac{1}{p}}. \end{aligned} \tag{4.165}$$

By Hölder inequality with $1/p+1/q=1$,

$$\begin{aligned} I_2 &\leqslant\int_{\mathbb{R}_+}\left\|\frac{1}{s-z}\right\|_{L^q(\mathbb{R}_+)}\||s|^{-1/2}\|_{L^p(\mathbb{R}_+)}e^{-ctv}dv\\ &\lesssim\int_{\mathbb{R}_+}|v-y|^{1/q-1}v^{-\frac{1}{2}+\frac{1}{p}}e^{-ctv}dv\\ &\lesssim t^{-\frac{1}{2}}. \end{aligned} \tag{4.166}$$

For $z \in \Omega_{0\pm k}$, $k = 1, 2$, we can conclude that $\|S\|_{L^\infty(\mathbb{C})\to L^\infty(\mathbb{C})} \lesssim t^{-\frac{1}{2}}$.

For $z \in \Omega_{11}$, we make $z = x + iy$, $s = \xi_1 + u + iv$. Thanks to (4.159), we have

$$\frac{1}{\pi}\iint_{\Omega_{11}} \frac{|W^{(3)}(s)|}{|s-z|} dA(s) \lesssim \frac{1}{\pi}\iint_{\Omega_{11}} \frac{|\bar{\partial} f_{11}|e^{-2t\Im\theta}}{|s-z|} dA(s). \tag{4.167}$$

Recalling Proposition 4.9, we still divide the integral into two parts

$$I_3 = \iint_{\Omega_{11}} \frac{|p'_{11}(s)|e^{-2t\Im\theta}}{|s-z|} dA(s), \quad I_4 = \iint_{\Omega_{11}} \frac{|s-\xi_1|^{-\frac{1}{2}} e^{-2t\Im\theta}}{|s-z|} dA(s). \tag{4.168}$$

For I_3, we use Proposition 4.7

$$\begin{aligned}
I_3 &= \int_0^\infty \int_{\frac{v}{\tan\phi}}^\infty \frac{|p'_{11}|e^{-ctv(\Re s-\xi_1)}}{|s-z|} dudv \\
&\lesssim \int_0^\infty \int_v^\infty \frac{|p'_{11}|e^{-ctvu}}{|s-z|} dudv \\
&\lesssim \int_0^\infty e^{-ctv^2} \|p'_{11}\|_{L^2(v,+\infty)} \left\| \frac{1}{s-z} \right\|_{L^2(v,+\infty)} dv \\
&\overset{(4.160)}{\lesssim} \int_0^{+\infty} e^{-4tv^2} |v-y|^{-1/2} dv \lesssim t^{-1/4}.
\end{aligned} \tag{4.169}$$

For I_4, we have

$$\begin{aligned}
I_4 &\lesssim \int_0^\infty e^{-ctv^2} dv \int_v^{+\infty} \frac{|s-\xi_2|^{-\frac{1}{2}}}{|s-z|} du \\
&\leqslant \int_0^{+\infty} e^{-ctv^2} \left\| \frac{1}{s-z} \right\|_{L^q(\mathbb{R}_+)} \left\| |s-\xi_1|^{-\frac{1}{2}} \right\|_{L^p(\mathbb{R}_+)} dv \\
&= \left(\int_0^y + \int_y^{+\infty} \right) v^{-\frac{1}{2}+\frac{1}{p}} |v-y|^{1/q-1} e^{-ctv^2} dv.
\end{aligned} \tag{4.170}$$

For the first integral, we have

$$\int_0^y v^{-\frac{1}{2}+\frac{1}{p}}|v-y|^{1/q-1}e^{-ctv^2}dv = \int_0^1 \sqrt{y}e^{-cty^2w^2}w^{1/p-1/2}|1-w|^{1/q-1}$$
$$\lesssim t^{-\frac{1}{4}}. \tag{4.171}$$

For the second integral, taking $v = y + w$, we have

$$\begin{aligned}\int_y^{+\infty} e^{-ctv^2}v^{\frac{1}{p}-\frac{1}{2}}|v-y|^{1/q-1}dv &= \int_0^{+\infty} e^{-t(y+w)^2}(y+w)^{1/p-1/2}w^{1/q-1}dw \\ &\leqslant \int_0^{+\infty} e^{-ctw^2}w^{1/p-1/2}w^{1/q-1}dw \\ &= \int_0^{+\infty} e^{-tw^2}w^{-\frac{1}{2}}dw \lesssim t^{-\frac{1}{4}}. \end{aligned}\tag{4.172}$$

For $z \in \Omega_{jk}$, $j = 1, 4$, $k = 1, 2, 3, 4$, we can conclude that $\|S\|_{L^\infty(\mathbb{C})\to L^\infty(\mathbb{C})} \lesssim t^{-\frac{1}{4}}$.

For $z \in \Omega_{24}$, we make $z = x + iy$, $s = \xi_2 + u + iv$. Since Ω_{24} is a bounded domain, we find that $0 < u < \frac{\xi_1-\xi_2}{2}$, $0 < v < \frac{(\xi_1-\xi_2)\tan\phi}{2}$ $(< \frac{\xi_1-\xi_2}{2})$. Owing to (4.159), we have

$$\frac{1}{\pi}\iint_{\Omega_{24}} \frac{|W^{(3)}(s)|}{|s-z|}dA(s) \lesssim \frac{1}{\pi}\iint_{\Omega_{24}} \frac{\langle s\rangle|\bar{\partial} f_{24}e^{-2it\theta}|}{|s-z||s-1|}dA(s) = \frac{1}{\pi}\iint_{\Omega_{24}} \frac{\langle s\rangle|\bar{\partial} f_{24}|e^{2t\Im\theta}}{|s-z||s-1|}dA(s). \tag{4.173}$$

Furthermore,

$$\frac{1}{\pi}\iint_{\Omega_{24}} \frac{\langle s\rangle|\bar{\partial} f_{24}|e^{2t\Im\theta}}{|s-z||s-1|}dA(s) \lesssim I_5 + I_6, \tag{4.174}$$

where

$$I_5 = \frac{1}{\pi}\iint_{\Omega_{24}} \frac{\langle s\rangle|\bar{\partial} f_{24}|e^{2t\Im\theta}\chi_{[\xi_2,1)}(|s|)}{|s-z||s-1|}dA(s), \tag{4.175}$$

$$I_6 = \frac{1}{\pi}\iint_{\Omega_{24}} \frac{\langle s\rangle|\bar{\partial} f_{24}|e^{2t\Im\theta}\chi_{\left[1,\frac{\xi_1-\xi_2}{\sqrt{2}}\right)}(|s|)}{|s-z||s-1|}dA(s), \tag{4.176}$$

where $\chi_{[\xi_2,1)}(|s|) + \chi_{\left[1,\frac{\xi_1-\xi_2}{\sqrt{2}}\right)}(|s|)$ is the partition of unity.

We consider I_5 firstly. Since $|s| \in [\xi_2, 1)$, we have $\frac{\langle s\rangle}{|s-1|} = \mathcal{O}(1)$. Recalling Proposition 4.9, we have

$$I_5 \leqslant I_5^{(1)} + I_5^{(2)}, \tag{4.177}$$

where

$$I_5^{(1)} = \iint\limits_{\Omega_{24}} \frac{|p'_{24}(s)| e^{2t\Im\theta}}{|s-z|} dA(s), \quad I_5^{(2)} = \iint\limits_{\Omega_{24}} \frac{|s-\xi_2|^{-\frac{1}{2}} e^{2t\Im\theta}}{|s-z|} dA(s). \tag{4.178}$$

Proposition 4.6 implies that

$$\begin{aligned} I_5^{(1)} &\leqslant \int_0^{+\infty} \int_{\frac{v}{\tan\phi}}^{+\infty} \frac{|p'_{24}(s)|}{|s-z|} e^{-ct(1+|s|^{-2})v^2} du dv \\ &\lesssim \int_{\mathbb{R}_+} \left\| \frac{1}{s-z} \right\|_{L^2(\mathbb{R}_+)} \|p'_{24}\|_{L^2(\mathbb{R}_+)} e^{-ctv^2} dv \\ &\lesssim \left(\int_0^y + \int_y^{+\infty} \right) |v-y|^{-\frac{1}{2}} e^{-ctv^2} dv. \end{aligned} \tag{4.179}$$

Then

$$\begin{aligned} \int_0^y (y-v)^{-\frac{1}{2}} e^{-ctv^2} dv &\lesssim \int_0^y (y-v)^{-\frac{1}{2}} v^{-\frac{1}{2}} dv \cdot t^{-\frac{1}{4}} \\ &\lesssim t^{-\frac{1}{4}}, \end{aligned} \tag{4.180}$$

where we use $e^{-z} \lesssim z^{-1/4}$. Setting $w = v - y$, we obtain

$$\int_y^{+\infty} (v-y)^{-\frac{1}{2}} e^{-ctv^2} dv = \int_0^{+\infty} w^{-\frac{1}{2}} e^{-ct(w+y)^2} dw \lesssim e^{-cty^2}. \tag{4.181}$$

Thus $I_5^{(1)} \lesssim t^{-1/4}$. We turn to estimate $I_5^{(2)}$. For $p > 2$, the similar analysis to (4.166) implies that

$$\begin{aligned} I_5^{(2)} &\leqslant \int_0^{+\infty} \left\| \frac{1}{s-z} \right\|_{L^q(\mathbb{R}_+)} \left\| |s-\xi_2|^{-\frac{1}{2}} \right\|_{L^p(\mathbb{R}_+)} e^{-ctv^2} dv \\ &\lesssim \int_0^{+\infty} v^{-\frac{1}{2}+\frac{1}{p}} |v-y|^{1/q-1} e^{-ctv^2} dv \end{aligned}$$

$$= \left(\int_0^y + \int_y^{+\infty} \right) v^{-\frac{1}{2}+\frac{1}{p}} |v - y|^{1/q-1} e^{-ctv^2} dv. \tag{4.182}$$

The analysis for $I_5^{(1)}$ can be applied to bound the two integrals above.

$$\begin{aligned} \int_0^y (y-v)^{\frac{1}{q}-1} v^{-\frac{1}{2}+\frac{1}{p}} e^{-ctv^2} dv &\lesssim t^{-\frac{1}{4}} \int_0^y (y-v)^{-\frac{1}{2}} dv \\ &\lesssim t^{-\frac{1}{4}}, \end{aligned} \tag{4.183}$$

and

$$\int_y^{+\infty} v^{-\frac{1}{2}+\frac{1}{p}} |v-y|^{1/q-1} e^{-ctv^2} dv \leqslant \int_0^{+\infty} w^{-\frac{1}{2}} e^{-ct(w+y)^2} dw \lesssim e^{-cty^2}. \tag{4.184}$$

Next we consider I_6. Since $|s| \in \left[1, \frac{\xi_1-\xi_2}{\sqrt{2}}\right)$, we have $\langle s \rangle \leqslant c(\xi)$ for $\xi \in \mathcal{R}_L$. The singularity at $z = 1$ could be balanced by (4.67),

$$\begin{aligned} I_6 &\leqslant c(\xi) \frac{1}{\pi} \iint_{\Omega_{24}} \frac{|p'_{24}| e^{2t\Im\theta}}{|s-z|} dA(s) \\ &\lesssim \int_0^{+\infty} \int_{\frac{v}{\tan\phi}}^{+\infty} \frac{|p'_{24}|}{|s-z|} e^{-ct(1+|s|^{-2})v^2} du dv \\ &\lesssim \int_{\mathbb{R}_+} \left\| \frac{1}{s-z} \right\|_{L^2(\mathbb{R}_+)} \|p'_{24}\|_{L^2(\mathbb{R}_+)} e^{-ctv^2} dv \\ &\lesssim \left(\int_0^y + \int_y^{+\infty} \right) |v-y|^{-\frac{1}{2}} e^{-ctv^2} dv \\ &\text{(similar to } I_5^{(1)}) \lesssim t^{-1/4}. \end{aligned} \tag{4.185}$$

From estimations for I_5 and I_6, we conclude that $\|S\|_{L^\infty(\mathbb{C}) \to L^\infty(\mathbb{C})} \lesssim t^{-\frac{1}{4}}$ for $z \in \Omega_{jk}$, $j = 2, 3$, $k = 1, 2, 3, 4$. Based on the three cases we discuss, $\|S\|_{L^\infty(\mathbb{C}) \to L^\infty(\mathbb{C})} \lesssim t^{-\frac{1}{4}}$ as $t \to \infty$. □

Following from Proposition 4.28, $(1 - S)^{-1}$ exists as $t \to \infty$. Finally, we turn to evaluate $M^{(3)}$ as $t \to \infty$. Make the asymptotic expansion as follows.

$$M^{(3)} = I + z^{-1} M_1^{(3)}(x,t) + \mathcal{O}(z^{-2}), \quad \text{as} \quad z \to \infty, \tag{4.186}$$

where

$$M_1^{(3)}(x,t) = \frac{1}{\pi} \iint_{\mathbb{C}} M^{(3)}(s) W^{(3)}(s) dA(s). \tag{4.187}$$

To recover the solution of defocusing mKdV (1.1), we shall discuss the asymptotic behavior of $M_1^{(3)}(x,t)$.

Proposition 4.29. *As $t \to \infty$ for $\xi \in \mathcal{R}_L$,*

$$|M_1^{(3)}(x,t)| \lesssim t^{-\frac{3}{4}}. \tag{4.188}$$

Proof. Noticing the boundedness of $M^{(PR)}$ and $M^{(3)}$, we have

$$\begin{aligned} |M_1^{(3)}| &\leqslant \iint_{\Omega_{jk}} \left| M^{(3)} M^{(PR)} \bar{\partial} R^{(2)} \left(M^{(PR)} \right)^{-1} \right| dA(s) \\ &\lesssim \iint_{\Omega_{jk}} \frac{\langle s \rangle}{|s-1|} \left| \bar{\partial} f_{jk} e^{\pm 2it\theta} \right| dA(s) \\ &= \iint_{\Omega_{jk}} \frac{\langle s \rangle}{|s-1|} \left| \bar{\partial} f_{jk} \right| e^{\mp 2t \Im\theta} dA(s) \end{aligned} \tag{4.189}$$

Similar to the proof of Proposition 4.28, we only take into account that matrix-valued functions have support in the sector Ω_{jk}. What we need to do is to evaluate the integral $\iint_{\Omega_{jk}} \frac{\langle s \rangle}{|s-1|} \left| \bar{\partial} f_{jk} \right| e^{\mp 2t\Im\theta} dA(s)$ for $z \in \Omega_{jk}$, $j = 0^{\pm}, 1, 2, 3, 4$, $k = 1, 2, 3, 4$. We exhibit details for $z \in \Omega_{0+1}$, $z \in \Omega_{11}$ and $z \in \Omega_{24}$. We point out that the analysis for $z \in \Omega_{24}$ is a bit different from $z \in \Omega_{0+1}, \Omega_{11}$, because we should deal with the singularity at $z = 1$ as what we do in the proof of Proposition 4.28.

For $z \in \Omega_{0+1}$, we make $z = x + iy$, $s = u + iv$ which satisfy $0 < x, u < \xi_2/2$, $0 < y, v < \frac{\xi_2 \tan\phi}{2}$ $(< \frac{\xi_2}{2})$. Owe to (4.159), $\langle s \rangle / |s-1| = \mathcal{O}(1)$ for $z \in \Omega_{0+1}$. And we can divide the integral into two parts

$$\iint_{\Omega_{0+1}} \left| \bar{\partial} f_{0+1}(s) \right| e^{-2t\Im\theta} dA(s) \lesssim I_1 + I_2, \tag{4.190}$$

where

$$I_1 = \iint_{\Omega_{0+1}} |p'_{0+1}(s)| e^{-2t\Im\theta} dA(s), \tag{4.191}$$

$$I_2 = \iint_{\Omega_{0+1}} |s|^{-\frac{1}{2}} e^{-2t\Im\theta} dA(s). \tag{4.192}$$

Since $\|p'_{0+1}\|_{L^2}$ is bounded, we bound I_1 by Cauchy-Schwarz inequality

$$\begin{aligned}
|I_1| &\leqslant \iint\limits_{\Omega_{0+1}} |p'_{0+1}| e^{-2ctv} dA(s) \\
&\leqslant \int_0^{\frac{\xi_2 \tan\phi}{2}} \|p'_{0+1}\|_{L^2(v,\xi_2/2)} \left(\int_v^{\xi_2/2} e^{-4ctv} du \right)^{1/2} dv \\
&\lesssim \int_0^{+\infty} v^{\frac{1}{2}} e^{-2ctv} dv \\
&\overset{w=v^{\frac{1}{2}}}{=} 2 \int_0^{+\infty} w^2 e^{-2ctw^2} dw \lesssim t^{-\frac{3}{2}}.
\end{aligned} \tag{4.193}$$

For I_2, we use Hölder inequality and (4.165) to obtain

$$\begin{aligned}
|I_2| &\leqslant \int_0^{\frac{\xi_2 \tan\phi}{2}} \| |s|^{-\frac{1}{2}} \|_{L^p(v,\xi_2/2)} \left(\int_v^{\xi_2/2} e^{-2ctqv} du \right)^{1/q} dv \\
&\lesssim \int_0^{+\infty} v^{-\frac{1}{2}+\frac{1}{p}} v^{\frac{1}{q}} e^{-2ctv} dv \\
&\overset{\frac{1}{p}+\frac{1}{q}=1}{=} \int_0^{+\infty} v^{\frac{1}{2}} e^{-2ctv} dv \\
&\lesssim t^{-\frac{3}{2}}.
\end{aligned} \tag{4.194}$$

For $z \in \Omega_{0\pm k}$, $k = 1, 2$, we conclude that $|M_1^{(3)}| \lesssim t^{-\frac{3}{2}}$.

For $z \in \Omega_{11}$, $\langle s \rangle / |s-1| = \mathcal{O}(1)$. And we make $z = x + iy$, $s = \xi_1 + u + iv$.

$$\iint\limits_{\Omega_{11}} \left| \bar{\partial} f_{11}(s) \right| e^{-2t\Im\theta} dA(s) \lesssim I_3 + I_4, \tag{4.195}$$

where

$$I_3 = \iint\limits_{\Omega_{11}} |p'_{11}(s)| e^{-2t\Im\theta} dA(s), \tag{4.196}$$

$$I_4 = \iint\limits_{\Omega_{11}} |s - \xi_1|^{-\frac{1}{2}} e^{-2t\Im\theta} dA(s). \tag{4.197}$$

With the help of Proposition 4.7, we can bound I_3, I_4. Using Cauchy-Schwarz inequality,

$$\begin{aligned}
|I_3| &\leqslant \iint_{\Omega_{11}} |p_{11}'(s)| e^{-2tv(\Re s-\xi_1)} dA(s) \\
&= \iint_{\Omega_{11}} |p_{11}'(s)| e^{-2tvu} dA(s) \\
&\overset{r\in H^1}{\leqslant} \int_0^{+\infty} \|p_{11}'(s)\|_{L^2(v+\xi_1,\infty)} \left(\int_v^{+\infty} e^{-4tuv} du \right)^{\frac{1}{2}} dv \\
&\lesssim t^{-\frac{1}{2}} \int_0^{+\infty} v^{-\frac{1}{2}} e^{-2tv^2} \lesssim t^{-\frac{3}{4}}.
\end{aligned} \tag{4.198}$$

As for I_4, we take the advantage of Hölder inequality and (4.165) again

$$\begin{aligned}
|I_4| &\leqslant \int_0^{+\infty} v^{-\frac{1}{2}+\frac{1}{p}} \left(\int_v^{+\infty} e^{-2tquv} du \right)^{\frac{1}{q}} dv \\
&\lesssim \int_0^{+\infty} v^{-\frac{1}{2}+\frac{1}{p}} (qtv)^{-\frac{1}{q}} e^{-2tv^2} dv \\
&\overset{\frac{1}{p}+\frac{1}{q}=1}{\lesssim} t^{-\frac{1}{q}} \int_0^{+\infty} v^{2/p-3/2} e^{-2tv^2} dv \\
&= t^{-\frac{3}{4}} \int_0^{+\infty} w^{\frac{2}{p}-\frac{3}{2}} e^{-2w^2} dw \\
&\lesssim t^{-\frac{3}{4}},
\end{aligned} \tag{4.199}$$

where we use the substitution $w = t^{1/2} v$. For $z \in \Omega_{jk}$, $j = 1, 4$, $k = 1, 2, 3, 4$, we can conclude that $|M_1^{(3)}| \lesssim t^{-\frac{3}{4}}$.

For $z \in \Omega_{24}$, we make $z = x + iy$, $s = \xi_2 + u + iv$ which satisfy $0 < u < \frac{\xi_1-\xi_2}{2}$, $0 < v < \frac{\xi_1-\xi_2}{2}\tan\phi$ $(< \frac{\xi_1-\xi_2}{2})$.

$$\iint_{\Omega_{24}} \left|\bar{\partial} f_{24}(s)\right| e^{2t\Im\theta} dA(s) \lesssim I_5 + I_6, \tag{4.200}$$

where

$$I_5 = \frac{1}{\pi} \iint_{\Omega_{24}} \frac{\langle s\rangle |\bar{\partial} f_{24}| e^{2t\Im\theta} \chi_{[\xi_2,1)}(|s|)}{|s-1|} dA(s), \tag{4.201}$$

$$I_6 = \frac{1}{\pi} \iint_{\Omega_{24}} \frac{\langle s \rangle |\bar{\partial} f_{24}| e^{2t\Im\theta} \chi_{\left[1, \frac{\xi_1 - \xi_2}{\sqrt{2}}\right)}(|s|)}{|s-1|} dA(s), \tag{4.202}$$

and $\chi_{[\xi_2,1)}(|s|) + \chi_{\left[1, \frac{\xi_1 - \xi_2}{\sqrt{2}}\right)}(|s|)$ is the partition of unity.

Notice $\langle s \rangle / |s-1| = \mathcal{O}(1)$ for $|s| \in [\xi_2, 1)$. Combining Proposition 4.9, we divide I_5 into two parts

$$I_5 \lesssim I_5^{(1)} + I_5^{(2)}, \tag{4.203}$$

where

$$I_5^{(1)} = \iint_{\Omega_{24}} |p_{24}'(s)| e^{2t\Im\theta} dA(s), \tag{4.204}$$

$$I_5^{(2)} = \iint_{\Omega_{24}} |s - \xi_2|^{-\frac{1}{2}} e^{2t\Im\theta} dA(s). \tag{4.205}$$

With the help of Proposition 4.6, we can bound $I_5^{(1)}$, $I_5^{(2)}$. We bound $I_5^{(1)}$ by Cauchy-Schwarz inequality

$$\begin{aligned}
|I_5^{(1)}| &\leqslant \iint_{\Omega_{24}} |p_{24}'(s)| e^{-2c(1+|z|^{-2})tv^2} dA(s) \\
&\leqslant \iint_{\Omega_{24}} |p_{24}'(s)| e^{-2ctv^2} dA(s) \\
&\leqslant \int_0^{\frac{\xi_1 - \xi_2}{2}\tan\phi} \|p_{24}'(s)\|_{L^2(v+\xi_2, \frac{\xi_1-\xi_2}{2})} \left(\int_v^{\frac{\xi_1 - \xi_2}{2}} e^{-4ctv^2} du \right)^{1/2} dv \\
&\lesssim \int_0^{+\infty} v^{\frac{1}{2}} e^{-2ctv^2} dv \\
&\overset{w = t^{\frac{1}{4}} v^{\frac{1}{2}}}{=} t^{-\frac{3}{4}} \int_0^{+\infty} w^2 e^{-2cw^4} dw \\
&\lesssim t^{-\frac{3}{4}}.
\end{aligned} \tag{4.206}$$

For $I_5^{(2)}$, the Hölder inequality and (4.165) are used to obtain

$$
\begin{aligned}
|I_5^{(2)}| &\leqslant \int_0^{\frac{\xi_1-\xi_2}{2}\tan\phi} \||s-\xi_2|^{-\frac{1}{2}}\|_{L^p(v+\xi_2,\frac{\xi_1-\xi_2}{2})} \left(\int_v^{\frac{\xi_1-\xi_2}{2}} e^{-2ctqv^2} du \right)^{1/q} dv \\
&\lesssim \int_0^{+\infty} v^{-\frac{1}{2}+\frac{1}{p}} v^{\frac{1}{q}} e^{-2ctv^2} dv \\
&\overset{\frac{1}{p}+\frac{1}{q}=1}{=} \int_0^{+\infty} v^{\frac{1}{2}} e^{-2ctv^2} dv \\
&\lesssim t^{-\frac{3}{4}}.
\end{aligned} \tag{4.207}
$$

We finally deal with I_6. Thanks to (4.67), the singularity at $z = 1$ can be balanced. Additionally, for $|s| \in \left[1, \frac{\xi_1-\xi_2}{\sqrt{2}}\right)$, $\langle s\rangle \leqslant c(\xi)$. As a consequence,

$$
\begin{aligned}
|I_6| &\leqslant c(\xi) \iint_{\Omega_{24}} |p_{24}'| e^{2t\Im\theta} dA(s) \\
&\lesssim \int_0^{+\infty} \int_{\frac{v}{\tan\phi}}^{+\infty} |p_{24}'| e^{-ct(1+|s|^{-2})v^2} du dv \\
&\leqslant \int_0^{+\infty} \|p_{24}'\|_{L^2(\mathbb{R}_+)} \left(\int_v^{+\infty} e^{-2ctv^2} du \right)^{\frac{1}{2}} dv \\
&\lesssim \int_0^{+\infty} v^{\frac{1}{2}} e^{-ctv^2} dv \lesssim t^{-3/4}.
\end{aligned} \tag{4.208}
$$

Summarizing the estimations for I_5 and I_6, we conclude that $|M_1^{(3)}| \lesssim t^{-\frac{3}{4}}$ for $z \in \Omega_{jk}$, $j = 2, 3$, $k = 1, 2, 3, 4$.

In a conclusion, $|M_1^{(3)}| \lesssim t^{-\frac{3}{2}} + t^{-\frac{3}{4}} + t^{-\frac{3}{4}} \lesssim t^{-\frac{3}{4}}$ as $t \to \infty$. $\square$

4.6. *Proof of Theorem 1.3(a)*

Reviewing the transformation (4.15), (4.73), (4.129) and (4.147):

$$
M^{(1)} = MG\delta^{\sigma_3}, \quad M^{(2)} = M^{(1)}R^{(2)}, \quad M^{(3)} = M^{(2)}(M^{(PR)})^{-1}, \quad M^{(PR)} = M^{(err)}M^{(\infty)}, \tag{4.209}
$$

we have

$$
M(z) = M^{(3)}(z)M^{(err)}(z)M^{(\infty)}(z)R^{(2)}(z)^{-1}\delta(z)^{-\sigma_3}G(z)^{-\sigma_3}, \quad z \in \mathbb{C}\backslash U(\xi). \tag{4.210}
$$

Taking $z \to \infty$ out $\overline{\Omega}$ ($R^{(2)} = I$, $G(z) = I$), we obtain

$$M(z) = \left(I + z^{-1}M_1^{(3)} + \cdots\right)\left(I + z^{-1}M_1^{(err)} + \cdots\right)\left(I + z^{-1}M_1^{(\infty)} + \cdots\right)$$
$$\times \left(I - z^{-1}\delta_1\sigma_3 + \cdots\right), \tag{4.211}$$

thus

$$M_1 = M_1^{(\infty)} + M_1^{(err)} + M_1^{(3)} - \delta_1\sigma_3. \tag{4.212}$$

Using the formulae (2.49), we have

$$q(x,t) = -i\left(M_1^{(\infty)}\right)_{12} - i\left(M_1^{(err)}\right)_{12} + \mathcal{O}\left(t^{-\frac{3}{4}}\right). \tag{4.213}$$

Combining Proposition 4.17 and Proposition 4.26, Theorem 1.3(a) follows.

5. Asymptotics for $\xi \in \mathcal{R}_R$: right field

5.1. First transformation: $M \to M^{(1)}$

By the Fig. 4(c), the jump factorization

$$V(z) = \begin{pmatrix} 1 & 0 \\ \frac{r(z)e^{-2it\theta}}{1-|r(z)|^2} & 1 \end{pmatrix}\left(1 - |r(z)|^2\right)^{\sigma_3}\begin{pmatrix} 1 & -\frac{\overline{r(z)}e^{2it\theta}}{1-|r(z)|^2} \\ 0 & 1 \end{pmatrix}, \quad z \in \Sigma,$$

plays a key role in our analysis. Therefore, we choose

$$\delta(z) := \delta(z,\xi) = \exp\left[-\frac{1}{2\pi i}\int_{\Sigma} \log\left(1 - |r(s)|^2\right)\frac{1}{s-z}ds\right]. \tag{5.1}$$

Remark 5.1. The difference between the $\delta(z)$ defined by (4.4) and the $\delta(z)$ defined by (5.1) is the integral interval. The interval of the former is $\Gamma := (-\infty, \xi_4) \cup (\xi_3, 0) \cup (0, \xi_2) \cup (\xi_1, +\infty)$, however, the latter is Σ.

Define

$$M^{(1)}(z) = M(z)G(z)\delta(z)^{\sigma_3}, \tag{5.2}$$

with

$$G(z)=\begin{cases}\begin{pmatrix}1 & -\frac{z-\eta_n}{c_n e^{-2it\theta(\eta_n)}}\\ 0 & 1\end{pmatrix}, & z\in D(\eta_n,h),\\ \begin{pmatrix}1 & 0\\ -\frac{z-\bar{\eta}_n}{\bar{c}_n e^{2it\theta(\bar{\eta}_n)}} & 1\end{pmatrix}, & z\in D(\bar{\eta}_n,h),\\ I & elsewhere.\end{cases} \tag{5.3}$$

The jump matrix $V^{(1)}(z)$ is as follows

$$V^{(1)}(z)=\begin{cases}\begin{pmatrix}1 & 0\\ \frac{r(z)\delta_-^2(z)}{1-|r(z)|^2}e^{-2it\theta} & 1\end{pmatrix}\begin{pmatrix}1 & -\frac{\overline{r(z)}\delta_+^{-2}(z)}{1-|r(z)|^2}e^{2it\theta}\\ 0 & 1\end{pmatrix}, & z\in\Sigma,\\ \begin{pmatrix}1 & -\frac{z-\eta_n}{c_n\delta^2 e^{-2it\theta(\eta_n)}}\\ 0 & 1\end{pmatrix}, & z\in\partial D(\eta_n,h) \text{ oriented counterclockwise},\\ \begin{pmatrix}1 & 0\\ \frac{z-\bar{\eta}_n}{\bar{c}_n\delta^{-2}(z)e^{2it\theta(\bar{\eta}_n)}} & 1\end{pmatrix}, & z\in\partial D(\bar{\eta}_n,h) \text{ oriented clockwise}.\end{cases} \tag{5.4}$$

The asymptotic behavior of $M^{(1)}$ is the same as the former section.

5.2. Opening $\bar{\partial}$ lenses: $M^{(1)}\to M^{(2)}$

Find a small sufficiently angle $\phi:\phi<\theta_0$ and define a new region $\Omega=\cup_{j=1,2,3,4}\Omega_j$, where

$$\Omega_1=\{z\in\mathbb{C}:0<\arg z<\phi\}, \qquad \Omega_2=\{z\in\mathbb{C}:\pi-\phi<\arg z<\phi\}, \tag{5.5}$$

$$\Omega_3=\{z\in\mathbb{C}:-\pi<\arg z<-\pi+\phi\}, \qquad \Omega_4=\{z\in\mathbb{C}:-\phi<\arg z<0\}. \tag{5.6}$$

Some paths are denoted by

$$\Sigma_1=e^{i\phi}\mathbb{R}_+, \qquad \Sigma_2=e^{i(\pi-\phi)}\mathbb{R}_+, \tag{5.7}$$

$$\Sigma_3=e^{-i(\pi-\phi)}\mathbb{R}_+, \qquad \Sigma_4=e^{-i\phi}\mathbb{R}_+, \tag{5.8}$$

with the left-to-right oriented boundaries of Ω, see Fig. 11.

Proposition 5.2. *For $\xi\in\mathcal{R}_R$, $z=le^{i\phi}$, and $F(l)=l+l^{-1}$, the phase function $\theta(z)$ defined by* (3.1) *satisfies*

$$\Im\theta(z)\geqslant\frac{1}{2}F(l)|\sin\phi|[\xi+F^2(l)], \quad z\in\Omega_j, \quad j=1,2, \tag{5.9}$$

$$\Im\theta(z)\leqslant-\frac{1}{2}F(l)|\sin\phi|[\xi+F^2(l)], \quad z\in\Omega_j, \quad j=3,4. \tag{5.10}$$

Proof. We give the details for $z\in\Omega_1$, the proof for the other regions is similar. Recalling (4.30), we have

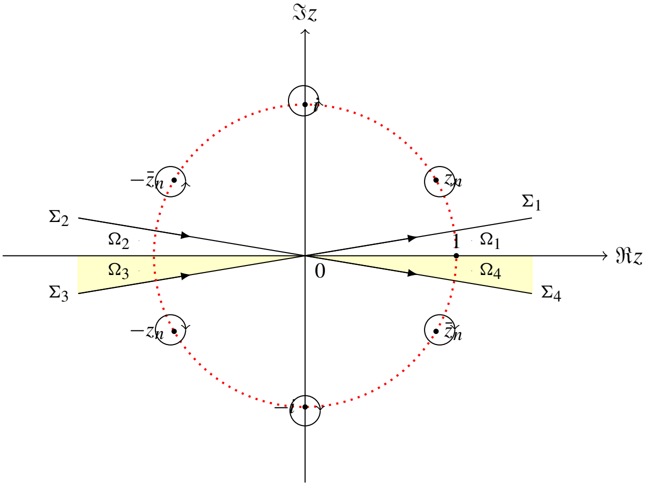


Fig. 11. For $\xi \in \mathcal{R}_R$, there are no phase points on the jump contour. The white regions imply that $e^{2it\theta} \to 0$, however, the yellow regions imply that $e^{-2it\theta} \to 0$.

$$\Im\theta(z) = \frac{1}{2}F(l)\sin\phi\left[\left(2F^2(l)-6\right)\cos(2\phi)+\xi+F^2(l)\right]$$
$$\overset{F(l)\geqslant 2}{\geqslant} \frac{1}{2}F(l)\sin\phi[\xi+F^2(l)] > 0. \quad \square \tag{5.11}$$

We choose $R^{(2)}(z)$ as

$$R^{(2)}(z) = \begin{cases} \begin{pmatrix} 1 & f_j e^{2it\theta} \\ 0 & 1 \end{pmatrix}, & z \in \Omega_j, \quad j = 1, 2, \\ \begin{pmatrix} 1 & 0 \\ f_j e^{-2it\theta} & 1 \end{pmatrix} & z \in \Omega_j, \quad j = 3, 4, \\ I, & elsewhere, \end{cases} \tag{5.12}$$

where f_j is given by the following proposition.

Proposition 5.3 *(Opening lens at $z = 0$ for $\xi \in \mathcal{R}_R$). $f_j : \overline{\Omega}_j \to \mathbb{C}$, $j = 1, 2, 3, 4$ are continuous on $\overline{\Omega}_j$ with boundary values:*

$$f_j(z) = \begin{cases} \dfrac{\overline{r(z)}}{1-|r(z)|^2}\delta_+^{-2}(z), & z \in \mathbb{R}, \\ 0, & z \in \Sigma_j, \quad j = 1, 2, \end{cases} \tag{5.13}$$

$$f_j(z) = \begin{cases} \dfrac{r(z)}{1-|r(z)|^2}\delta_-^{2}(z), & z \in \mathbb{R}, \\ 0, & z \in \Sigma_j, \quad j = 3, 4. \end{cases} \tag{5.14}$$

And f_j, $j = 1, 2, 3, 4$ have following estimation:

$$|\bar{\partial} f_j(z)| \leqslant c|z|^{-\frac{1}{2}} + c|r'(|z|)| + c\varphi(|z|), \qquad j=1,2,3,4, \tag{5.15}$$

where $\varphi \in C_0^\infty(\mathbb{R},[0,1])$ *is a cutoff function with small support near 1.*

Moreover

$$|\bar{\partial} f_j(z)| \leqslant c|z-1|, \quad z \in \Omega_j, \quad j=1,4, \tag{5.16}$$

$$|\bar{\partial} f_j(z)| \leqslant c|z+1|, \quad z \in \Omega_j, \quad j=2,3. \tag{5.17}$$

Proof. The proof is an analogue of [35, Proposition 5.5] or [10, Lemma 6.5]. A sketch proof for f_1 is exhibited as follows. By $r(z) \to \mp i$ as $z \to \pm 1$, we know that $|r(z)| \to 1$ as $z \to \pm 1$. This implies that $f_1(z)$ is singular at $z=1$. However, the singular behavior is exactly balanced by the factor $\delta^2(z)$. With the help of (2.19)-(2.21), we have

$$\frac{\overline{r(z)}}{1-|r(z)|^2}\delta_+^{-2}(z) = \frac{\overline{b(z)}}{a(z)}\left(\frac{a(z)}{\delta_+(z)}\right)^2 = \frac{\overline{J_b(z)}}{J_a(z)}\left(\frac{a(z)}{\delta_+(z)}\right)^2, \tag{5.18}$$

where $J_a(z)=\det(\Phi_{+,1},\Phi_{-,2})$, $J_b(z)=\det(\Phi_{-,1},\Phi_{+,1})$. It's not difficult to know that the denominator of each factor in the r.h.s. of (5.18) is nonzero and analytic in Ω_1, with a well defined nonzero limit on $\partial\Omega_1$. Notice also that in Ω_1 away from the point $z=1$ the factors in the l.h.s. of (5.18) are well behaved.

We introduce the cutoff functions $\chi_0, \chi_1 \in C_0^\infty(\mathbb{R},[0,1])$ with small support near $z=0$ and $z=1$ respectively, such that for any sufficiently small s, $\chi_0(s)=\chi_1(s+1)=1$. Additionally, we impose the condition $\chi_1(s)=\chi_1(s^{-1})$ to preserve symmetry. Then we can rewrite the function $f_1(z)$ in $\mathbb{R}_+$ as $f_1(z)=f_1^{(1)}(z)+f_1^{(2)}(z)$, where

$$f_1^{(1)}(z) = (1-\chi_1(z))\frac{\overline{r(z)}}{1-|r(z)|^2}\delta_+^{-2}(z), \qquad f_1^{(2)}(z) = \chi_1(z)\frac{\overline{J_b(z)}}{J_a(z)}\left(\frac{a(z)}{\delta_+(z)}\right)^2. \tag{5.19}$$

The aim of (5.19) is to balance the effect raised by the singularity $z=1$ due to $|r(1)|=1$. Fix a small $\kappa_0>0$, we extend $f_1^{(1)}(z)$ and $f_1^{(2)}(z)$ in Ω_1 by

$$f_1^{(1)}(z) = (1-\chi_1(|z|))\frac{\overline{r(|z|)}}{1-|r(|z|)|^2}\delta^{-2}(z)\cos(\kappa \arg z), \tag{5.20}$$

$$f_1^{(2)}(z) = h(|z|)g(z)\cos(\kappa\arg z) + \frac{i|z|}{\kappa}\chi_0\left(\frac{\arg z}{\kappa_0}\right)h'(|z|)g(z)\sin(\kappa\arg z), \tag{5.21}$$

where

$$\kappa := \frac{\pi}{2\theta_0}, \quad h(z) := \chi_1(z)\frac{\overline{J_b(z)}}{J_a(z)}, \quad g(z) := \left(\frac{a(z)}{\delta(z)}\right)^2. \tag{5.22}$$

Notice that the definition of f_1 preserves the symmetry $f_1(s) = -\overline{f_1(\bar{s}^{-1})}$.

Firstly we bound the $\bar{\partial} f_1^{(1)}$.

$$\bar{\partial} f_1^{(1)}(z) = -\frac{\bar{\partial}\chi_1(|z|)}{\delta^2(z)} \frac{\overline{r(|z|)}\cos(\kappa \arg z)}{1-|r(|z|)|^2} + \frac{1-\chi_1(|z|)}{\delta^2(z)} \bar{\partial} \left(\frac{\overline{r(|z|)}\cos(\kappa \arg z)}{1-|r(|z|)|^2} \right). \tag{5.23}$$

We know that $1-|r(z)|^2 > c > 0$ as $z \in \text{supp}(1-\chi_1(|z|))$ and $\delta^{-2}(z)$ is bounded as $z \in \Omega_1 \cap \text{supp}(1-\chi_1(|z|))$. Taking $z = le^{i\gamma} := u + iv$, we still have the equality $\bar{\partial} = \frac{e^{i\gamma}}{2}(\partial_l + il^{-1}\partial_\gamma)$ and apply it to the first term of (5.23)

$$\left| -\frac{\bar{\partial}\chi_1(|z|)}{\delta^2(z)} \frac{\overline{r(|z|)}\cos(\kappa \arg z)}{1-|r(|z|)|^2} \right| = \left| \frac{e^{i\gamma}\chi_1' \bar{r} \cos(\kappa\gamma)}{2\delta^2(z)(1-|r(|z|)|^2)} \right| \lesssim \varphi(|z|) \tag{5.24}$$

for a appropriate $\varphi \in C_0^\infty(\mathbb{R}, [0, 1])$ with a small support near 1 and with $\varphi = 1$ on $\text{supp}\chi_1$. As $r(0) = 0$ and $r(z) \in H^1(\mathbb{R})$ it follows that $|r(|z|)| \leqslant |z|^{1/2}\|r'\|_{L^2(\mathbb{R})}$, we have

$$\left| \frac{1-\chi_1(|z|)}{\delta^2(z)} \bar{\partial} \left(\frac{\overline{r(|z|)}\cos(\kappa \arg z)}{1-|r(|z|)|^2} \right) \right| \lesssim |r'(z)| + \frac{|r(z)|}{|z|} \lesssim |r'(z)| + |z|^{-\frac{1}{2}}. \tag{5.25}$$

So we obtain that

$$|f_1^{(1)}(z)| \lesssim \varphi(|z|) + |r'(z)| + |z|^{-\frac{1}{2}}. \tag{5.26}$$

Next we bound $\bar{\partial} f_1^{(2)}$,

$$\begin{aligned} \bar{\partial} f_1^{(2)} = \frac{1}{2} e^{i\gamma} g(z) \Bigg[& h' \cos(\kappa\gamma) \left(1 - \chi_0\left(\frac{\gamma}{\kappa_0}\right)\right) - \frac{i\kappa h(l)}{l} \sin(\kappa\gamma) \\ & + \frac{i}{\kappa}(lh'(l))' \sin(\kappa\gamma) \chi_0\left(\frac{\gamma}{\kappa_0}\right) + \frac{i}{\kappa\kappa_0} h'(l) \sin(\kappa\gamma) \chi_0'\left(\frac{\gamma}{\kappa_0}\right) \Bigg] \end{aligned} \tag{5.27}$$

in which $g(z)$ is bounded, $q \in L^{1,2}(\mathbb{R})$ and $q' \in W^{1,1}(\mathbb{R})$. So we claim that $\bar{\partial} f_1^{(2)}(z) \lesssim \varphi(|z|)$ for a $\varphi \in C_0^\infty(\mathbb{R}, [0, 1])$ with a small support near 1, thus yielding (5.15).

Finally, as $z \to 1$, we have

$$|\bar{\partial} f_1^{(2)}(z)| = \mathcal{O}(\gamma) \tag{5.28}$$

from which (5.16) follows immediately. $\square$

We now use $R^{(2)}$ to define transformation $M^{(2)} = M^{(1)} R^{(2)}$, which help us set up the following mixed $\bar{\partial}$-RH problem for $\xi \in \mathcal{R}_R$:

RH problem 5.4. *Find a 2×2 matrix-valued function $M^{(2)}(z)$ such that*
- $M^{(2)}(z)$ is continuous in $\mathbb{C} \backslash \Sigma^{(2)}$, where $\Sigma^{(2)} = \Sigma^{pole} \cup (\cup_{j=1,2,3,4} \Sigma_j)$ (see Fig. 11).

- $M^{(2)}(z)$ takes continuous boundary values $M^{(2)}_{\pm}(z)$ on $\Sigma^{(2)}$ with jump relation

$$M^{(2)}_{+}(z) = M^{(2)}_{-}(z)V^{(2)}(z), \tag{5.29}$$

where $V^{(2)} = I$ for $z \in \Sigma_j$, $j = 1, 2, 3, 4$.
- Asymptotic behavior

$$M^{(2)}(x, t; z) = I + \mathcal{O}(z^{-1}), \quad z \to \infty, \tag{5.30}$$

$$M^{(2)}(x, t; z) = \frac{\sigma_2}{z} + \mathcal{O}(1), \quad z \to 0. \tag{5.31}$$

- For $z \in \mathbb{C}$, we have $\bar{\partial}$-derivative equality

$$\bar{\partial} M^{(2)} = M^{(2)} \bar{\partial} R^{(2)}, \tag{5.32}$$

where

$$\bar{\partial} R^{(2)} = \begin{cases} \begin{pmatrix} 1 & \bar{\partial} f_j e^{2it\theta} \\ 0 & 1 \end{pmatrix}, & z \in \Omega_j, \quad j = 1, 2 \\ \begin{pmatrix} 1 & 0 \\ \bar{\partial} f_j e^{-2it\theta} & 1 \end{pmatrix} & z \in \Omega_j, \quad j = 3, 4, \\ 0, & elsewhere. \end{cases} \tag{5.33}$$

5.3. Analysis on pure $\bar{\partial}$ problem

The error order mainly comes from the $\bar{\partial}$-problem for $\xi \in \mathcal{R}_R$, which is different from the previous Section 4. We focus our insights on the estimates for the Cauchy-Green operator S defined by (4.153) and $M_1^{(3)}(x, t)$ defined by (4.187). Then we have the following two propositions.

Proposition 5.5. *Consider the operator S defined in (4.153), then we have $S : L^\infty(\mathbb{C}) \to L^\infty(\mathbb{C}) \cap C^0(\mathbb{C})$ and*

$$\|S\|_{L^\infty(\mathbb{C}) \to L^\infty(\mathbb{C})} \lesssim t^{-\frac{1}{2}}. \tag{5.34}$$

Proof. The proof is an analogue of Proposition 4.28. □

Proposition 5.6. *As $t \to \infty$ for $\xi \in \mathcal{R}_R$,*

$$|M_1^{(3)}(x, t)| \lesssim t^{-1}, \quad \text{as} \quad t \to \infty. \tag{5.35}$$

Proof. We present the details for $z \in \Omega_1$. By the standard procedure, as Proposition 4.29, we have

$$|M_1^{(3)}(x, t)| \lesssim I_1 + I_2 + I_3, \tag{5.36}$$

where

$$I_1 = \iint_{\Omega_1} \frac{\langle s\rangle |\bar{\partial} f_1| e^{-2t\Im\theta} \chi_{[0,1)}(|s|)}{s-1} dA(s), \tag{5.37}$$

$$I_2 = \iint_{\Omega_1} \frac{\langle s\rangle |\bar{\partial} f_1| e^{-2t\Im\theta} \chi_{[1,2)}(|s|)}{s-1} dA(s), \tag{5.38}$$

$$I_3 = \iint_{\Omega_1} \frac{\langle s\rangle |\bar{\partial} f_1| e^{-2t\Im\theta} \chi_{[2,+\infty)}(|s|)}{s-1} dA(s). \tag{5.39}$$

For the term with $\chi_{[2,+\infty)}(|s|)$ the factor $\langle s\rangle |s-1|^{-1} = \mathcal{O}(1)$, and fixing a $p > 2$, $q \in (1,2)$ we obtain the superabound for I_3

$$\begin{aligned} I_3 &\lesssim \iint \left[|r'(|s|)| + \varphi(|s|) + |s|^{-\frac{1}{2}} \right] e^{-2t\Im\theta} \chi_{[1,+\infty)}(|s|) dA(s) \\ &\lesssim \int_{\mathbb{R}_+} \|e^{-ctuv}\|_{L^2(\max\{v,1/\sqrt{2}\},\infty)} + \|e^{-ctuv}\|_{L^2(\max\{v,1/\sqrt{2}\},\infty)} \||s|^{-1/2}\|_{L^q(v,\infty)} dv \\ &\lesssim \int_{\mathbb{R}_+} e^{-ctv} \left((tv)^{-1/2} + t^{-1/p} v^{-1/p+1/q-1/2} \right) dv \lesssim t^{-1}. \end{aligned} \tag{5.40}$$

For $s \in [0,2)$, $\langle s\rangle \leqslant 5$, so it could be omitted from the remaining estimates. For the $\chi_{[1,2]}(|s|)$, we use (5.15) to obtain that $I_2 \lesssim t^{-1}$ at once. For the $\chi_{[0,1)}(|s|)$, the changes of variables $w = \bar{z}^{-1}$ and $r(s) = -\overline{r(\bar{s}^{-1})}$ imply that

$$I_1 = \iint_{\Omega_1} |\bar{\partial} f_1| e^{-2t\Im\theta(w)} |w-1|^{-1} \chi_{[1,\infty]}(|w|) |w|^{-1} dA(s) \lesssim t^{-1}. \tag{5.41}$$

Finally, we get the desired estimate. □

5.4. *Proof of Theorem 1.3(c)*

Similar to the Subsection 4.6,

$$M(z) = M^{(3)}(z) M^{(\infty)}(z) R^{(2)}(z)^{-1} \delta(z)^{-\sigma_3}, \tag{5.42}$$

for z outside $\overline{\Omega}$.

Furthermore, we obtain

$$M_1 = M_1^{(\infty)} + M_1^{(3)} - \delta_1 \sigma_3, \tag{5.43}$$

which yields Theorem 1.3(c) by using the formulae (2.49).

Data availability

No data was used for the research described in the article.

Acknowledgments

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Appendix A. Parabolic cylinder model near ξ_j, $j = 1, 2, 3, 4$

This appendix is based on the methods developed by A. Its' fundamental work [17].

A.1. Local model near ξ_j, $j = 1, 3$

We take ξ_1 as an example to present this standard model.

RH problem A.1. *Find a matrix-valued function $M^{(PC,\xi_1)}(\zeta) := M^{(PC,\xi_1)}(\zeta;\xi)$ such that*
- *$M^{(PC,\xi_1)}(\zeta;\xi)$ is analytical in $\mathbb{C}\setminus\Sigma^{pc}$ with $\Sigma^{pc} = \left\{\mathbb{R}e^{i\phi}\right\} \cup \left\{\mathbb{R}e^{i(\pi-\phi)}\right\}$ shown in Fig. A.1.*
- *$M^{(PC,\xi_1)}$ has continuous boundary values $M_\pm^{(PC,\xi_1)}$ on Σ^{pc} and*

$$M_+^{(PC,\xi_1)}(\zeta) = M_-^{(PC,\xi_1)}(\zeta)V^{pc}(\zeta), \quad \zeta \in \Sigma^{pc}, \tag{A.1}$$

where

$$V^{pc}(\zeta) = \begin{cases} \zeta^{-i\nu\hat{\sigma}_3} e^{\frac{i\zeta^2}{4}\hat{\sigma}_3} \begin{pmatrix} 1 & -\frac{\bar{r}_{\xi_1}}{1-|r_{\xi_1}|^2} \\ 0 & 1 \end{pmatrix}, & \zeta \in \mathbb{R}_+ e^{\phi i}, \\ \zeta^{-i\nu\hat{\sigma}_3} e^{\frac{i\zeta^2}{4}\hat{\sigma}_3} \begin{pmatrix} 1 & 0 \\ \frac{r_{\xi_1}}{1-|r_{\xi_1}|^2} & 1 \end{pmatrix}, & \zeta \in \mathbb{R}_+ e^{(2\pi-\phi)i}, \\ \zeta^{-i\nu\hat{\sigma}_3} e^{\frac{i\zeta^2}{4}\hat{\sigma}_3} \begin{pmatrix} 1 & -\bar{r}_{\xi_1} \\ 0 & 1 \end{pmatrix}, & \zeta \in \mathbb{R}_+ e^{(\pi+\phi)i}, \\ \zeta^{-i\nu\hat{\sigma}_3} e^{\frac{i\zeta^2}{4}\hat{\sigma}_3} \begin{pmatrix} 1 & 0 \\ r_{\xi_1} & 1 \end{pmatrix}, & \zeta \in \mathbb{R}_+ e^{(\pi-\phi)i}, \end{cases} \tag{A.2}$$

with $\nu = \nu(\xi_1)$.
- *Asymptotic behavior:*

$$M^{(PC,\xi_1)}(\zeta) = I + M_1^{(PC,\xi_1)}\zeta^{-1} + \mathcal{O}(\zeta^{-2}), \qquad \zeta \to \infty. \tag{A.3}$$

The RHP A.1 has an explicit solution, which can be expressed in terms of Webber equation $(\frac{\partial^2}{\partial z^2} + (\frac{1}{2} - \frac{z^2}{2} + a))D_a(z) = 0$. Taking the transformation

$$M^{(PC,\xi_1)} = \psi(\zeta)\mathcal{P}\zeta^{i\nu\sigma_3}e^{-\frac{i}{4}\zeta^2\sigma_3}, \tag{A.4}$$

where

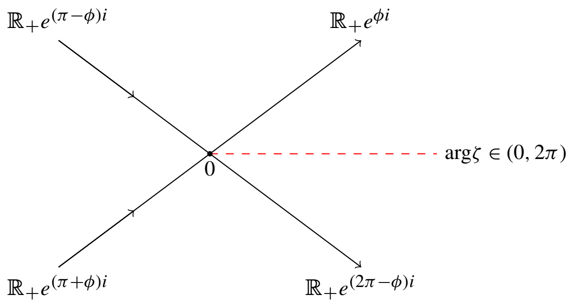

Fig. A.1. The contour Σ^{pc} for the case of ξ_j, $j = 1, 3$.

$$\mathcal{P}(\xi) = \begin{cases} \begin{pmatrix} 1 & \frac{\bar{r}_{\xi_1}}{1-|r_{\xi_1}|^2} \\ 0 & 1 \end{pmatrix}, & \arg\zeta \in (0, \phi), \\ \begin{pmatrix} 1 & 0 \\ \frac{r_{\xi_1}}{1-|r_{\xi_1}|^2} & 1 \end{pmatrix}, & \arg\zeta \in (2\pi - \phi, 2\pi), \\ \begin{pmatrix} 1 & 0 \\ -r_{\xi_1} & 1 \end{pmatrix}, & \arg\zeta \in (\pi - \phi, \pi), \\ \begin{pmatrix} 1 & -\bar{r}_{\xi_1} \\ 0 & 1 \end{pmatrix}, & \arg\zeta \in (\pi, \pi + \phi), \\ I, & else. \end{cases} \tag{A.5}$$

The function ψ satisfies the following RH conditions.

RH problem A.2. *Find a* 2×2 *matrix-valued function* $\psi(\zeta)$ *such that*
- ψ *is analytical in* $\mathbb{C}\backslash\mathbb{R}$*;*
- Due to the branch cut along $\mathbb{R}_+$*,* $\psi(\zeta)$ *takes continuous boundary values* $\psi_\pm$ *on* $\mathbb{R}$ *and*

$$\psi_+(\zeta) = \psi_-(\zeta)V^\psi, \quad \zeta \in \mathbb{R}, \tag{A.6}$$

where

$$V^\psi(\xi) = \begin{pmatrix} 1 - |r_{\xi_1}|^2 & -\bar{r}_{\xi_1} \\ r_{\xi_1} & 1 \end{pmatrix}. \tag{A.7}$$

- Asymptotic behavior:

$$\psi = \zeta^{-i\nu\sigma_3} e^{\frac{i}{4}\zeta^2\sigma_3} \left(I + M_1^{(PC,\xi_1)}\zeta^{-1} + \mathcal{O}(\zeta^{-2}) \right), \quad \text{as} \quad \zeta \to \infty. \tag{A.8}$$

Differentiating (A.6) with respect to ζ, and combining $\frac{i\zeta}{2}\sigma_3\psi_+ = \frac{i\zeta}{2}\sigma_3\psi_- V^\psi$, we obtain

$$\left(\frac{d\psi}{d\zeta} - \frac{i\zeta}{2}\sigma_3\psi \right)_+ = \left(\frac{d\psi}{d\zeta} - \frac{i\zeta}{2}\sigma_3\psi \right)_- V^\psi. \tag{A.9}$$

Notice that $\det V^{\psi} = 1$, thus we have $\det\psi_+ = \det\psi_-$. Moreover, we know that $\det\psi$ is holomorphic in $\mathbb{C}$ by Painlevé analytic continuation theorem. It follows that ψ^{-1} exists and is bounded. The matrix function $\left(\frac{d\psi}{d\zeta} - \frac{i\zeta}{2}\sigma_3\psi\right)\psi^{-1}$ has no jump along the real axis and is an entire function with respect to ζ. Combining (A.4), we can directly calculate that

$$\left(\frac{d\psi}{d\zeta} - \frac{i\zeta}{2}\sigma_3\psi\right)\psi^{-1} = \left[\frac{dM^{(PC,\xi_1)}}{d\zeta} - M^{(PC,\xi_1)}\frac{i\nu}{\zeta}\sigma_3\right]\left(M^{(PC,\xi_1)}\right)^{-1} + \frac{i\zeta}{2}\left[M^{(PC,\xi_1)},\sigma_3\right]\left(M^{(PC,\xi_1)}\right)^{-1}. \tag{A.10}$$

The first term in the R.H.S. of (A.10) tends to zero as $\zeta \to \infty$. We use $M^{(PC,\xi_1)}(\zeta) = I + M_1^{(PC,\xi_1)}\zeta^{-1} + \mathcal{O}(\zeta^{-2})$ as well as Liouville theorem to obtain that there exists a constant matrix β_1^{mat} such that

$$\begin{pmatrix} 0 & \beta_{12}^{(\xi_1)} \\ \beta_{21}^{(\xi_1)} & 0 \end{pmatrix} = \beta_1^{mat} = \frac{i}{2}\left[M_1^{(PC,\xi_1)},\sigma_3\right] = \begin{pmatrix} 0 & -i[M_1^{(PC,\xi_1)}]_{12} \\ i[M_1^{(PC,\xi_1)}]_{21} & 0 \end{pmatrix}, \tag{A.11}$$

which implies that $[M_1^{(PC,\xi_1)}]_{12} = i\beta_{12}^{(\xi_1)}$, $[M_1^{(PC,\xi_1)}]_{21} = -i\beta_{21}^{(\xi_1)}$. Using Liouville theorem again, we have

$$\left(\frac{d\psi}{d\zeta} - \frac{i\zeta}{2}\sigma_3\psi\right) = \beta_1^{mat}\psi. \tag{A.12}$$

Rewrite the above equality to the following ODE systems

$$\frac{d\psi_{11}}{d\zeta} - \frac{i\zeta}{2}\psi_{11} = \beta_{12}^{(\xi_1)}\psi_{21}, \tag{A.13}$$

$$\frac{d\psi_{21}}{d\zeta} + \frac{i\zeta}{2}\psi_{21} = \beta_{21}^{(\xi_1)}\psi_{11}, \tag{A.14}$$

as well as

$$\frac{d\psi_{12}}{d\zeta} - \frac{i\zeta}{2}\psi_{12} = \beta_{12}^{(\xi_1)}\psi_{22}, \tag{A.15}$$

$$\frac{d\psi_{22}}{d\zeta} + \frac{i\zeta}{2}\psi_{22} = \beta_{21}^{(\xi_1)}\psi_{12}. \tag{A.16}$$

From (A.13) to (A.16), we solve that

$$\frac{d^2\psi_{11}}{d\zeta^2} + \left(-\frac{i}{2} + \frac{\zeta^2}{4} - \beta_{12}^{(\xi_1)}\beta_{21}^{(\xi_1)}\right)\psi_{11} = 0, \quad \frac{d^2\psi_{21}}{d\zeta^2} + \left(\frac{i}{2} + \frac{\zeta^2}{4} - \beta_{12}^{(\xi_1)}\beta_{21}^{(\xi_1)}\right)\psi_{21} = 0, \tag{A.17}$$

$$\frac{d^2\psi_{12}}{d\zeta^2} + \left(-\frac{i}{2} + \frac{\zeta^2}{4} - \beta_{12}^{(\xi_1)}\beta_{21}^{(\xi_1)}\right)\psi_{12} = 0, \quad \frac{d^2\psi_{22}}{d\zeta^2} + \left(\frac{i}{2} + \frac{\zeta^2}{4} - \beta_{12}^{(\xi_1)}\beta_{21}^{(\xi_1)}\right)\psi_{22} = 0. \tag{A.18}$$

The Webber equation is

$$y'' + \left(\frac{1}{2} - \frac{z^2}{4} + a\right) y = 0. \tag{A.19}$$

The parabolic cylinder functions $D_a(z)$, $D_a(-z)$, $D_{-a-1}(iz)$, $D_{-a-1}(-iz)$ all satisfy (A.19) and are entire $\forall a$. The large z behavior of $D_a(z)$ can be uniquely given by the following formulaes.

$$D_a(z) = \begin{cases} z^a e^{-\frac{z^2}{4}}\left(1+\mathcal{O}(z^{-2})\right), \quad |\arg z| < \dfrac{3\pi}{4}, \\ z^a e^{-\frac{z^2}{4}}\left(1+\mathcal{O}(z^{-2})\right) - \dfrac{\sqrt{2\pi}}{\Gamma(-a)} e^{ia\pi} z^{-a-1} e^{z^2/4}\left(1+\mathcal{O}(z^{-2})\right), \quad \dfrac{\pi}{4} < \arg z < \dfrac{5\pi}{4}, \\ z^a e^{-\frac{z^2}{4}}\left(1+\mathcal{O}(z^{-2})\right) - \dfrac{\sqrt{2\pi}}{\Gamma(-a)} e^{-ia\pi} z^{-a-1} e^{z^2/4}\left(1+\mathcal{O}(z^{-2})\right), \\ \quad -\dfrac{5\pi}{4} < \arg z < -\dfrac{\pi}{4}. \end{cases}$$

We set $\nu = \beta_{12}^{(\xi_1)}\beta_{21}^{(\xi_1)}$. For ψ_{11}, $\Im\zeta > 0$, we introduce a new variable $\eta = \zeta e^{-\frac{i\pi}{4}}$, and the first equation of (A.17) becomes

$$\frac{d^2\psi_{11}}{d\eta^2} + \left(\frac{1}{2} - \frac{\eta^2}{4} - i\nu\right)\psi_{11} = 0. \tag{A.20}$$

For $\zeta \in \mathbb{C}_+$, $0 < \mathrm{Arg}\zeta < \pi$, $-\frac{\pi}{4} < \mathrm{Arg}\eta < \frac{3\pi}{4}$. We have $\psi_{11} = e^{\frac{\pi}{4}\nu(\xi_1)} D_{-i\nu(\xi_1)}(e^{-\frac{\pi}{4}i}\zeta) \sim \zeta^{-i\nu} e^{\frac{i}{4}\zeta^2}$, which corresponds to the $(1,1)$-entry of (A.8).

To limit the length of paper, we present the other results for ψ below without delicate calculation. The unique solution to RH problem A.2 is
when $\zeta \in \mathbb{C}_+$,

$$\psi(\zeta) = \begin{pmatrix} e^{\frac{\pi}{4}\nu(\xi_1)} D_{-i\nu(\xi_1)}(e^{-\frac{\pi}{4}i}\zeta) & \frac{i\nu(\xi_1)}{\beta_{21}^{(\xi_1)}} e^{-\frac{3\pi}{4}(\nu(\xi_1)+i)} D_{i\nu(\xi_1)-1}(e^{-\frac{3\pi i}{4}}\zeta) \\ -\frac{i\nu(\xi_1)}{\beta_{12}^{(\xi_1)}} e^{\frac{\pi}{4}(\nu(\xi_1)-i)} D_{-i\nu(\xi_1)-1}(e^{-\frac{\pi i}{4}}\zeta) & e^{-\frac{3\pi}{4}\nu(\xi_1)} D_{i\nu(\xi_1)}(e^{-\frac{3\pi}{4}i}\zeta) \end{pmatrix}, \tag{A.21}$$

when $\zeta \in \mathbb{C}_-$,

$$\psi(\zeta) = \begin{pmatrix} e^{\frac{5\pi}{4}\nu(\xi_1)} D_{-i\nu(\xi_1)}(e^{-\frac{5\pi}{4}i}\zeta) & \frac{i\nu(\xi_1)}{\beta_{21}^{(\xi_1)}} e^{-\frac{7\pi}{4}(\nu(\xi_1)+i)} D_{i\nu(\xi_1)-1}(e^{\frac{-7\pi i}{4}}\zeta) \\ -\frac{i\nu(\xi_1)}{\beta_{12}^{(\xi_1)}} e^{\frac{5\pi}{4}(\nu(\xi_1)-i)} D_{-i\nu(\xi_1)-1}(e^{-\frac{5\pi i}{4}}\zeta) & e^{-\frac{7}{4}\pi\nu(\xi_1)} D_{i\nu(\xi_1)}(e^{-\frac{7}{4}\pi i}\zeta) \end{pmatrix}. \tag{A.22}$$

Which is similar to [24, Appendix C.3].

From (A.6), we know that $(\psi_-)^{-1}\psi_+ = V^\psi$ and

$$
\begin{aligned}
r_{\xi_1} &= \psi_{-,11}\psi_{+,21} - \psi_{-,21}\psi_{+,11}\\
&= e^{\frac{5\pi}{4}\nu(\xi_1)}D_{-i\nu(\xi_1)}(e^{-\frac{5\pi}{4}i}\zeta)\cdot\frac{e^{\frac{\pi\nu(\xi_1)}{4}}}{\beta_{12}^{(\xi_1)}}\left[\partial_\zeta(D_{-i\nu(\xi_1)}(e^{-\frac{\pi i}{4}}\zeta)) - \frac{i\zeta}{2}D_{-i\nu(\xi_1)}(e^{-\frac{\pi i}{4}}\zeta)\right]\\
&\quad - e^{\frac{\pi}{4}\nu(\xi_1)}D_{-i\nu(\xi_1)}(e^{-\frac{\pi}{4}i}\zeta)\cdot\frac{e^{\frac{5\pi\nu(\xi_1)}{4}}}{\beta_{12}^{(\xi_1)}}\left[\partial_\zeta(D_{-i\nu(\xi_1)}(e^{-\frac{5\pi i}{4}}\zeta)) - \frac{i\zeta}{2}D_{-i\nu(\xi_1)}(e^{-\frac{5\pi i}{4}}\zeta)\right]\\
&= \frac{e^{\frac{3\pi}{2}\nu(\xi_1)}}{\beta_{12}^{(\xi_1)}}\mathrm{Wr}\left(D_{-i\nu(\xi_1)}(e^{-\frac{5\pi}{4}i}\zeta), D_{-i\nu(\xi_1)}(e^{-\frac{\pi}{4}i}\zeta)\right)\\
&= \frac{e^{\frac{3\pi}{2}\nu(\xi_1)}}{\beta_{12}^{(\xi_1)}}\cdot\frac{\sqrt{2\pi}e^{-\frac{5\pi}{4}i}}{\Gamma(i\nu(\xi_1))}.
\end{aligned}
\tag{A.23}
$$

The second "=" we use the equality $D_a'(z) + \frac{z}{2}D_a(z) = aD_{a-1}(z)$. As for the last "=", we use the Wronskian identity $\mathrm{Wr}(D_a(z), D_a(-z)) = \frac{\sqrt{2\pi}}{\Gamma(-a)}$.

And

$$
\beta_{12}^{(\xi_1)} = \frac{\sqrt{2\pi}e^{-\frac{5i\pi}{4}}e^{\frac{3\pi\nu(\xi_1)}{2}}}{r_{\xi_1}\Gamma(i\nu(\xi_1))}, \tag{A.24}
$$

$$
\beta_{12}^{(\xi_1)}\beta_{21}^{(\xi_1)} = \nu(\xi_1), \tag{A.25}
$$

$$
\begin{aligned}
\arg\beta_{21}^{(\xi_1)} &= -\frac{5\pi}{4} - \arg r_{\xi_1} - \arg\Gamma(i\nu(\xi_1))\\
&= -\frac{5\pi}{4} - 2\beta(\xi_1,\xi) + 2t\theta(\xi_1) + \nu(\xi_1)\log(2t\theta''(\xi_1)) - \arg\Gamma(i\nu(\xi_1)).
\end{aligned}
\tag{A.26}
$$

Finally we have

$$
M^{(PC,\xi_1)} = I + \frac{1}{\zeta}\begin{pmatrix} 0 & i\beta_{12}^{(\xi_1)} \\ -i\beta_{21}^{(\xi_1)} & 0 \end{pmatrix} + \mathcal{O}(\zeta^{-2}). \tag{A.27}
$$

The results of Appendix A.1 can also be applied to the local model near ξ_3.

A.2. Local model near ξ_j, $j = 2, 4$

RH problem A.3. *Find a matrix-valued function $M^{(PC,\xi_2)}(\zeta) := M^{(PC,\xi_2)}(\zeta;\xi)$ such that*
- $M^{(PC,\xi_2)}(\zeta;\xi)$ is analytical in $\mathbb{C}\setminus\Sigma^{pc}$ with $\Sigma^{pc} = \left\{\mathbb{R}e^{i\phi}\right\}\cup\left\{\mathbb{R}e^{i(\pi-\phi)}\right\}$ shown in Fig. A.2.
- $M^{(PC,\xi_2)}$ has continuous boundary values $M_\pm^{(PC,\xi_2)}$ on Σ^{pc} and

$$
M_+^{(PC,\xi_2)}(\zeta) = M_-^{(PC,\xi_2)}(\zeta)V^{pc}(\zeta), \quad \zeta\in\Sigma^{pc}, \tag{A.28}
$$

where

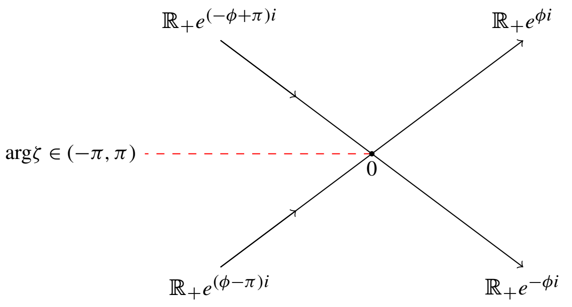

Fig. A.2. The contour Σ^{pc} for the case of ξ_j, $j = 2, 4$.

$$V^{pc}(\zeta) = \begin{cases} \zeta^{i\nu\hat{\sigma}_3} e^{-\frac{i\zeta^2}{4}\hat{\sigma}_3} \begin{pmatrix} 1 & 0 \\ r_{\xi_2} & 1 \end{pmatrix}, & \zeta \in \mathbb{R}_+e^{\phi i}, \\ \zeta^{i\nu\hat{\sigma}_3} e^{-\frac{i\zeta^2}{4}\hat{\sigma}_3} \begin{pmatrix} 1 & -\bar{r}_{\xi_2} \\ 0 & 1 \end{pmatrix}, & \zeta \in \mathbb{R}_+e^{-\phi i}, \\ \zeta^{i\nu\hat{\sigma}_3} e^{-\frac{i\zeta^2}{4}\hat{\sigma}_3} \begin{pmatrix} 1 & 0 \\ \frac{r_{\xi_2}}{1-|r_{\xi_2}|^2} & 1 \end{pmatrix}, & \zeta \in \mathbb{R}_+e^{(\phi-\pi)i}, \\ \zeta^{i\nu\hat{\sigma}_3} e^{-\frac{i\zeta^2}{4}\hat{\sigma}_3} \begin{pmatrix} 1 & -\frac{\bar{r}_{\xi_2}}{1-|r_{\xi_2}|^2} \\ 0 & 1 \end{pmatrix}, & \zeta \in \mathbb{R}_+e^{(-\phi+\pi)i}, \end{cases} \tag{A.29}$$

with $\nu = \nu(\xi_2)$.
- Asymptotic behavior:

$$M^{(PC,\xi_2)}(\zeta) = I + M_1^{(PC,\xi_2)}\zeta^{-1} + \mathcal{O}(\zeta^{-2}), \qquad \zeta \to \infty. \tag{A.30}$$

The RH problem A.3 has an explicit solution, which can be expressed in terms of Webber equation $(\frac{\partial^2}{\partial z^2} + (\frac{1}{2} - \frac{z^2}{2} + a))D_a(z) = 0$. Taking the transformation

$$M^{(PC,\xi_2)} = \psi(\zeta)\mathcal{P}\zeta^{-i\nu\sigma_3} e^{\frac{i}{4}\zeta^2\sigma_3}, \tag{A.31}$$

where

$$\mathcal{P}(\xi) = \begin{cases} \begin{pmatrix} 1 & 0 \\ -r_{\xi_2} & 1 \end{pmatrix}, & \arg\zeta \in (0,\phi), \\ \begin{pmatrix} 1 & -\bar{r}_{\xi_2} \\ 0 & 1 \end{pmatrix}, & \arg\zeta \in (-\phi, 0), \\ \begin{pmatrix} 1 & 0 \\ \frac{r_{\xi_2}}{1-|r_{\xi_2}|^2} & 1 \end{pmatrix}, & \arg\zeta \in (\phi-\pi, -\pi), \\ \begin{pmatrix} 1 & \frac{\bar{r}_{\xi_2}}{1-|r_{\xi_2}|^2} \\ 0 & 1 \end{pmatrix}, & \arg\zeta \in (-\phi+\pi, \pi), \\ I, & else. \end{cases} \tag{A.32}$$

The function ψ satisfies the following RH conditions.

RH problem A.4. *Find a 2×2 matrix-valued function $\psi(\zeta)$ such that*
- ψ is analytical in $\mathbb{C}\backslash\mathbb{R}$.
- Due to the branch cut along $\mathbb{R}_-$, $\psi(\zeta)$ takes continuous boundary values $\psi_\pm$ on $\mathbb{R}$ and

$$\psi_+(\zeta) = \psi_-(\zeta) V^{\psi}, \quad \zeta \in \mathbb{R}, \tag{A.33}$$

where

$$V^{\psi}(\xi) = \begin{pmatrix} 1 - |r_{\xi_2}|^2 & -\bar{r}_{\xi_2} \\ r_{\xi_2} & 1 \end{pmatrix}. \tag{A.34}$$

- Asymptotic behavior:

$$\psi = \zeta^{i\nu\sigma_3} e^{-\frac{i}{4}\zeta^2\sigma_3} \left(I + M_1^{(PC,\xi_2)} \zeta^{-1} + \mathcal{O}(\zeta^{-2}) \right), \quad \text{as} \quad \zeta \to \infty. \tag{A.35}$$

Differentiating (A.33) with respect to ζ, and combining $\frac{i\zeta}{2}\sigma_3\psi_+ = \frac{i\zeta}{2}\sigma_3\psi_- V^{\psi}$, we obtain

$$\left(\frac{d\psi}{d\zeta} + \frac{i\zeta}{2}\sigma_3\psi \right)_+ = \left(\frac{d\psi}{d\zeta} + \frac{i\zeta}{2}\sigma_3\psi \right)_- V^{\psi}. \tag{A.36}$$

Since the same reasons presented in Appendix A.1, the matrix function $\left(\frac{d\psi}{d\zeta} + \frac{i\zeta}{2}\sigma_3\psi \right)\psi^{-1}$ has no jump along the real axis and is an entire function with respect to ζ. Combining (A.31), we can directly calculate that

$$\begin{aligned} \left(\frac{d\psi}{d\zeta} + \frac{i\zeta}{2}\sigma_3\psi \right)\psi^{-1} = &\left[\frac{dM^{(PC,\xi_2)}}{d\zeta} + M^{(PC,\xi_2)} \frac{i\nu}{\zeta}\sigma_3 \right] \left(M^{(PC,\xi_2)} \right)^{-1} \\ &+ \frac{i\zeta}{2} \left[\sigma_3, M^{(PC,\xi_2)} \right] \left(M^{(PC,\xi_2)} \right)^{-1}. \end{aligned} \tag{A.37}$$

The first term in the R.H.S. of (A.37) tends to zero as $\zeta \to \infty$. We use $M^{(PC,\xi_2)}(\zeta) = I + M_1^{(PC,\xi_2)}\zeta^{-1} + \mathcal{O}(\zeta^{-2})$ as well as Liouville theorem to obtain that there exists a constant matrix β_2^{mat} such that

$$\begin{pmatrix} 0 & \beta_{12}^{(\xi_2)} \\ \beta_{21}^{(\xi_2)} & 0 \end{pmatrix} = \beta_2^{mat} = \frac{i}{2}\left[\sigma_3, M_1^{(PC,\xi_2)} \right] = \begin{pmatrix} 0 & i[M_1^{(PC,\xi_2)}]_{12} \\ -i[M_1^{(PC,\xi_2)}]_{21} & 0 \end{pmatrix}, \tag{A.38}$$

which implies that $[M_1^{(PC,\xi_2)}]_{12} = -i\beta_{12}^{(\xi_1)}$, $[M_1^{(PC,\xi_2)}]_{21} = i\beta_{21}^{(\xi_1)}$. Using Liouville theorem again, we have

$$\left(\frac{d\psi}{d\zeta} + \frac{i\zeta}{2}\sigma_3\psi \right) = \beta_2^{mat}\psi. \tag{A.39}$$

We rewrite the above equality to the following ODE system

$$\frac{d\psi_{11}}{d\zeta} + \frac{i\zeta}{2}\psi_{11} = \beta_{12}^{(\xi_2)}\psi_{21}, \tag{A.40}$$

$$\frac{d\psi_{21}}{d\zeta} - \frac{i\zeta}{2}\psi_{21} = \beta_{21}^{(\xi_2)}\psi_{11}, \tag{A.41}$$

as well as

$$\frac{d\psi_{12}}{d\zeta} + \frac{i\zeta}{2}\psi_{12} = \beta_{12}^{(\xi_2)}\psi_{22}, \tag{A.42}$$

$$\frac{d\psi_{22}}{d\zeta} - \frac{i\zeta}{2}\psi_{22} = \beta_{21}^{(\xi_2)}\psi_{12}. \tag{A.43}$$

From (A.40) to (A.43), we solve that

$$\frac{d^2\psi_{11}}{d\zeta^2} + \left(\frac{i}{2} + \frac{\zeta^2}{4} - \beta_{12}^{(\xi_2)}\beta_{21}^{(\xi_2)}\right)\psi_{11} = 0, \quad \frac{d^2\psi_{21}}{d\zeta^2} + \left(-\frac{i}{2} + \frac{\zeta^2}{4} - \beta_{12}^{(\xi_2)}\beta_{21}^{(\xi_2)}\right)\psi_{21} = 0, \tag{A.44}$$

$$\frac{d^2\psi_{12}}{d\zeta^2} + \left(\frac{i}{2} + \frac{\zeta^2}{4} - \beta_{12}^{(\xi_2)}\beta_{21}^{(\xi_2)}\right)\psi_{12} = 0, \quad \frac{d^2\psi_{22}}{d\zeta^2} + \left(-\frac{i}{2} + \frac{\zeta^2}{4} - \beta_{12}^{(\xi_2)}\beta_{21}^{(\xi_2)}\right)\psi_{22} = 0. \tag{A.45}$$

We set $\nu = \beta_{12}^{(\xi_2)}\beta_{21}^{(\xi_2)}$. For ψ_{11}, $\Im\zeta > 0$, we introduce the new variable $\eta = \zeta e^{-\frac{3i\pi}{4}}$, and the first equation of (A.44) becomes

$$\frac{d^2\psi_{11}}{d\eta^2} + \left(\frac{1}{2} - \frac{\eta^2}{4} + i\nu\right)\psi_{11} = 0. \tag{A.46}$$

For $\zeta \in \mathbb{C}_+$, $0 < \mathrm{Arg}\zeta < \pi$, $-\frac{3\pi}{4} < \mathrm{Arg}\eta < \frac{\pi}{4}$. We have $\psi_{11} = e^{-\frac{3\pi}{4}\nu(\xi_2)} D_{i\nu(\xi_2)}(e^{-\frac{3\pi}{4}i}\zeta) \sim \zeta^{i\nu} e^{-\frac{i}{4}\zeta^2}$, which corresponds to the (1, 1)-entry of (A.35). The other results for ψ are presented below.

The unique solution to RH problem A.4 is
when $\zeta \in \mathbb{C}_+$,

$$\psi(\zeta) = \begin{pmatrix} e^{-\frac{3\pi}{4}\nu(\xi_2)} D_{i\nu(\xi_2)}(e^{-\frac{3\pi}{4}i}\zeta) & -\frac{i\nu(\xi_2)}{\beta_{21}^{(\xi_2)}} e^{\frac{\pi}{4}(\nu(\xi_2)-i)} D_{-i\nu(\xi_2)-1}(e^{-\frac{\pi i}{4}}\zeta) \\ \frac{i\nu(\xi_2)}{\beta_{12}^{(\xi_2)}} e^{-\frac{3\pi}{4}(\nu(\xi_2)+i)} D_{i\nu(\xi_2)-1}(e^{-\frac{3\pi i}{4}}\zeta) & e^{\frac{\pi}{4}\nu(\xi_2)} D_{-i\nu(\xi_2)}(e^{-\frac{\pi}{4}i}\zeta) \end{pmatrix}, \tag{A.47}$$

when $\zeta \in \mathbb{C}_-$,

$$\psi(\zeta) = \begin{pmatrix} e^{\frac{\pi\nu(\xi_2)}{4}} D_{i\nu(\xi_2)}(e^{\frac{\pi}{4}i}\zeta) & -\frac{i\nu(\xi_2)}{\beta_{21}^{(\xi_2)}} e^{-\frac{3\pi}{4}(\nu(\xi_2)-i)} D_{-i\nu(\xi_2)-1}(e^{\frac{3\pi i}{4}}\zeta) \\ \frac{i\nu(\xi_2)}{\beta_{12}^{(\xi_2)}} e^{\frac{\pi}{4}(\nu(\xi_2)+i)} D_{i\nu(\xi_2)-1}(e^{\frac{\pi i}{4}}\zeta) & e^{-\frac{3\pi}{4}\nu(\xi_2)} D_{-i\nu(\xi_2)}(e^{\frac{3\pi}{4}i}\zeta) \end{pmatrix}. \tag{A.48}$$

Which is derived in [14, Section 4] and verified in [24, Proposition 5.5].

From (A.33), we know that $(\psi_-)^{-1}\psi_+ = V^{\psi}$ and

$$
\begin{aligned}
r_{\xi_2} &= \psi_{-,11}\psi_{+,21} - \psi_{-,21}\psi_{+,11} \\
&= e^{\frac{\pi}{4}\nu(\xi_2)} D_{i\nu(\xi_2)}(e^{\frac{\pi}{4}i}\zeta) \cdot \frac{e^{-\frac{3\pi\nu(\xi_2)}{4}}}{\beta_{12}^{(\xi_2)}} \left[\partial_\zeta (D_{i\nu(\xi_2)}(e^{-\frac{3\pi i}{4}}\zeta)) + \frac{i\zeta}{2} D_{i\nu(\xi_2)}(e^{-\frac{3\pi i}{4}}\zeta)\right] \\
&\quad - e^{-\frac{3\pi}{4}\nu(\xi_2)} D_{i\nu(\xi_2)}(e^{-\frac{3\pi}{4}i}\zeta) \cdot \frac{e^{\frac{\pi\nu(\xi_2)}{4}}}{\beta_{12}^{(\xi_2)}} \left[\partial_\zeta (D_{i\nu(\xi_2)}(e^{\frac{\pi i}{4}}\zeta)) + \frac{i\zeta}{2} D_{i\nu(\xi_2)}(e^{\frac{\pi i}{4}}\zeta)\right] \\
&= \frac{e^{-\frac{\pi}{2}\nu(\xi_2)}}{\beta_{12}^{(\xi_2)}} \mathrm{Wr}\left(D_{i\nu(\xi_2)}(e^{\frac{\pi}{4}i}\zeta), D_{i\nu(\xi_2)}(e^{-\frac{3\pi}{4}i}\zeta)\right) \\
&= \frac{e^{-\frac{\pi}{2}\nu(\xi_2)}}{\beta_{12}^{(\xi_2)}} \cdot \frac{\sqrt{2\pi}e^{\frac{\pi}{4}i}}{\Gamma(-i\nu(\xi_2))}.
\end{aligned} \tag{A.49}
$$

And

$$
\beta_{12}^{(\xi_2)} = \frac{\sqrt{2\pi}e^{\frac{i\pi}{4}}e^{-\frac{\pi\nu(\xi_2)}{2}}}{r_{\xi_2}\Gamma(-i\nu(\xi_2))}, \tag{A.50}
$$

$$
\beta_{12}^{(\xi_2)}\beta_{21}^{(\xi_2)} = \nu(\xi_2), \tag{A.51}
$$

$$
\begin{aligned}
\arg\beta_{12}^{(\xi_2)} &= \frac{\pi}{4} - \arg r_{\xi_2} - \arg\Gamma(-i\nu(\xi_2)) \\
&= \frac{\pi}{4} - 2\beta(\xi_2,\xi) + 2t\theta(\xi_2) - \nu(\xi_2)\log(-2t\theta''(\xi_2)) - \arg\Gamma(-i\nu(\xi_2)).
\end{aligned} \tag{A.52}
$$

As a consequence,

$$
M^{(PC,\xi_2)} = I + \frac{1}{\zeta}\begin{pmatrix} 0 & -i\beta_{12}^{(\xi_2)} \\ i\beta_{21}^{(\xi_2)} & 0 \end{pmatrix} + \mathcal{O}(\zeta^{-2}). \tag{A.53}
$$

The results of Appendix A.2 also can be applied to the local model near ξ_4.

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